WHEN ARE ALL CONTINUOUS FUNCTIONS TO Y CONSTANT?

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Translation of [Her] (H. Herrlich: Wann sind alle stetigen Abbildungen in Y konstant?).

Urysohn [U] asked whether for every regular space X (having at least two points) there is a non-constant continuous map from X do the space Y of real numbers. This question was negatively answered by Hewitt [Hew], Novák [N] and Van Est-Freudenthal [EF]. The methods used by these authors (which go back to Tychonoff [T]) let us show relatively easy the following result:

Theorem. *Let Y be a topological space. The following conditions are equivalent:*
(a) *Y is a T_1-space,*
(b) *there exists a regular space X (having at least two points), such that every continuous map from X to Y is constant.*

Proof. (b) $\Rightarrow$(a). If Y is not a T_1-space, then there are two different points in Y such that a is contained in every neighborhood of b. If X is a regular space with at least two points, B an open subset of X such that $B \neq \emptyset$, $B \neq X$, then the function from X to Y defined as

$$f(x) = \begin{cases} a & \text{for} x \in B, \\ b & \text{for} x \notin B \end{cases}$$

is continuous and not constant.

(a) $\Rightarrow$(b). Let Y be a T_1-space. The construction of the space X with the desired properties will be done in several steps. First we construct for $i = 1, 2$ the spaces R_i with a distinguished point r_i in a such way that every continuous map from R_i to Y is constant on some neighborhood of R_i. Then we show that the space T, which is obtained from the product $R_1 \times R_2$ by removing the point $(r_1; r_2)$, has the property, that for every continuous map f from T to Y there exist neighborhoods U_i of r_i such that f is constant on $U_1 \times U_2 - \{(r_1; r_2)\}$. Through gluing together countably many homeomorphic copies of the space T and addition of two points a, b we obtain a space Q with the property, that every continuous map from Q to Y has the same value at the point a as at the point b. With the help of Q we will show that every (regular) space Z is embeddable into a (regular) space $Q(Z)$ such that every continuous map from $Q(Z)$ to Y is constant on Z. This easily implies the existence of a space Z with the properties required in (b).

1. Construction of spaces R_i. Let $|Y| \leq \aleph_\alpha$ and for $i = 1, 2$ let R_i be a set with $|R_i| = \aleph_{\alpha+i}$ and let r_i be a fixed point from R_i. We declare a subset of R_i to be open if and only if $|R_i - B| < \aleph_{\alpha+1}$ whenever $r_i \in B$. Then R_i is a regular space with the property that the intersection of $\aleph_{\alpha+i-1}$ neighborhoods of r_i is again a neighborhood of r_i. If f is a continuous map from R_i to Y, then $f^{-1}(y_0) = \bigcap\{f^{-1}(Y - \{y\}); y \in Y - \{y_0\}\}$ again a neighborhood of r_i, since it is an

intersection of $\aleph_\alpha$ neighborhoods of r_i. Therefore f is constant on a neighborhood of r_i.

2. Proof of the properties of T. Let f be a continuous map from T to Y. Then for every $x \in R_1 - \{r_1\}$ there exists a neighborhood U_x of r_2 such that f is constant on $\{x\} \times U_x$. $V_2 = \bigcap\{U_x; x \in R_1 - \{r_1\}\}$ is a neighborhood or r_2. If y is an arbitrary element of $V_2 - \{r_2\}$, then there is a neighborhood V_1 of r_1, such that f is constant on $V_1 \times \{y\}$. Therefore f is constant on $(V_1 \times V_2) - \{(r_1; r_2)\}$.

3. Construction of the space Q. Let assign to every number n a space T_n, which is homeomorphic to T, and whose elements will be denoted by $(x; y; n)$. We add to the topological sum of the spaces T_n two points a, b and define the neighborhoods of a (resp. b) as all sets for which there exists an integer n with $\{\bigcup T_m; m \geq n\} \cup \{a\} \subseteq B$ (resp. $\{\bigcup T_m; m \geq n\} \cup \{b\} \subseteq B$), then we get a regular space R.

If we identify in R

1. for every even n and every $x \in R_1 - \{r_1\}$ the two points $(x; r_2; n)$ and $(x; r_2; n+1)$,

2. for every odd n and every $x \in R_2 - \{r_2\}$ the two points $(r_1; x; n)$ and $(r_1; x; n+1)$,

then we obtain a regular space Q with the property such that $f(a) = f(b)$ holds for every continuous map from Q to Y.

4. Construction of $Q(Z)$ for an arbitrary space Z. A subset B of $Z \times Q$ is open in $Z \times Q$ if and only if the following holds:

(1) If $(z; x)$ is an element of B, then there is a neighborhood U of x in Q with $\{z\} \times U \subset B$.
(2) If $(z; a)$ is an element of B, then there is a neighborhood U of z in Z with $U \times \{a\} \subset B$.

If we identify in the above space $Z \times Q$ all points of the set $Z \times \{b\}$, then we obtain a space $Q(Z)$. The assignment $h(z) = (z; a)$ defines a homeomorphic map h from Z into $Q(Z)$. So we can assume that Z is canonically embedded into $Q(Z)$. If f is a continuous map from $Q(Z)$ to Y, then f is constant on Z. If Z is regular, then so is $Q(Z)$.

5. Definition of X. Let X_0 be a singleton. By induction we define $X_{n+1} = Q(X_n)$. Then $X_0 \subset X_1 \subset X_2 \subset \dots$ are regular spaces. Let a subset of $X = \bigcup\{X_n; n = 0, 1, \dots\}$ be open if and only if $B \cap X_n$ is open for every n. The space X is a regular space. Every continuous map from X to Y is constant. □

Remark. *The above results has a trivial analogue:*
Let X be a topological space. The following conditions are equivalent:
(a) *X is connected,*
(b) *there is a regular space Y (having at least 2 points), such that every continuous map from X to Y is constant.*

References

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