CANTOR-BERNSTEINOVA VETA

$|X| = |Y| \iff$ u. $X \xrightarrow{bij} Y$

$|X| \leq |Y| \iff$ u. $X \xrightarrow{inj} Y$

Tvrdenie 3.1.5. *Nech* X, Y, Z *sú ľubovoľné množiny. Potom platí:*
(i) $|X| \leq |X|$;
(ii) $|X| \leq |Y| \wedge |Y| \leq |Z| \Rightarrow |X| \leq |Z|$
(iii) $|X| = |Y| \Rightarrow |X| \leq |Y|$

Dôkaz. (i) $id_X : X \to X$ je injekcia.
(ii) Zloženie dvoch injekcií je injekcia.
(iii) Každá bijekcia je injekcia.

$|X| \leq |Y| \wedge |Y| \leq |X| \Rightarrow |X| = |Y|$

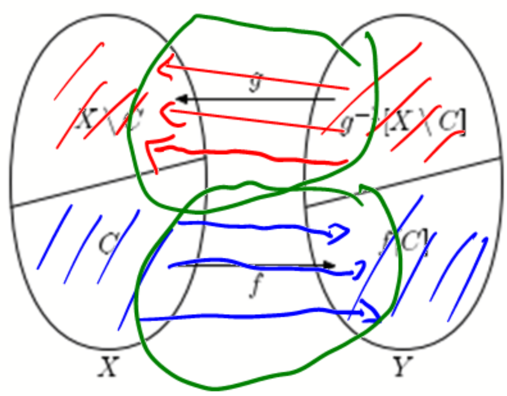
Veta 3.1.6 (Cantor-Bernstein). *Nech* X, Y *sú množiny. Ak platí* $|X| \leq |Y|$ *a* $|Y| \leq |X|$, *tak* $|X| = |Y|$.

$$|X| \leq |Y| \wedge |Y| \leq |X| \Rightarrow |X| = |Y|$$

Inak: Ak existuje injekcia $f : X \to Y$ *a injekcia* $g : Y \to X$, *tak existuje bijekcia* $h : X \to Y$.

$f : X \longrightarrow Y$

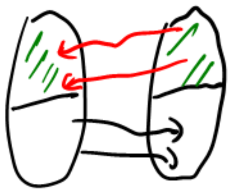
$X \xleftarrow{g} Y$

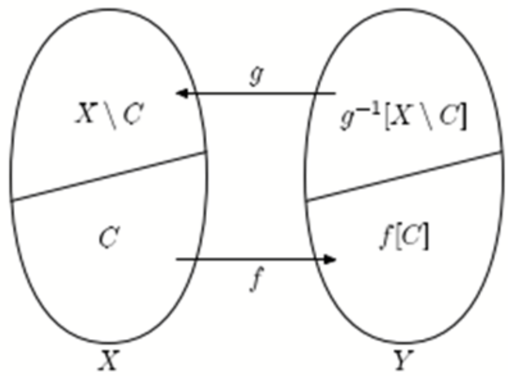


? $\exists$ takéto C ?

D:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$





$C \subseteq X$

$$g[Y \setminus f[C]] = X \setminus C$$

$$X \setminus g[Y \setminus f[C]] = C$$

$$F : \mathcal{P}(X) \longrightarrow \mathcal{P}(Y) \qquad \mathcal{P}(X) = \{A \subseteq X\}$$

$$F: \quad F(A) = X \setminus g[Y \setminus f[A]]$$

CHCEM: $(\exists C \subseteq X)\ \boxed{F(C) = C}$

M $\qquad$ M

$$A \subseteq B \quad \Rightarrow \quad F(A) \subseteq F(B).$$

$$(x \leq y \Rightarrow f(x) \leq f(y))$$

$$A \subseteq B$$

$$f[A] \subseteq f[B] \qquad 1$$

$$Y \setminus f[A] \supseteq Y \setminus f[B] \qquad 2$$

$$g[Y \setminus f[A]] \supseteq g[Y \setminus f[B]] \qquad 1$$

$$X \setminus g[Y \setminus f[A]] \subseteq X \setminus g[Y \setminus f[B]] \qquad 2$$

$$F(A) \subseteq F(B)$$

1 $A \subseteq B \Rightarrow f[A] \subseteq f[B]$

2 $A \subseteq B \Rightarrow C \setminus A \supseteq C \setminus B$

$$\mathcal{Y} = \{B \subseteq X;\ B \subseteq F(B)\}$$

$$C := \bigcup \mathcal{Y} = \bigcup \{B \subseteq X;\ B \subseteq F(B)\}$$

? $C = F(C)$?

(3) $(\forall B \in \mathcal{Y})\ B \subseteq \bigcup \mathcal{Y}$

(4) $(\forall B \in \mathcal{Y})\ B \subseteq D \Rightarrow \bigcup \mathcal{Y} \subseteq D$

$$(\forall B \in \mathcal{Y})\ B \subseteq C \overset{(M)}{\Longrightarrow} F(B) \subseteq F(C)$$

$$B \in \mathcal{Y} \longrightarrow B \subseteq F(B)$$

$$\Downarrow \quad \Downarrow$$

pro lib. $B \in \mathcal{Y}$ platí $B \subseteq F(C)$.

$$\bigcup_{B \in \mathcal{Y}} B = \bigcup \mathcal{Y} = \boxed{C \subseteq F(C)}$$

$$C \subseteq F(C) \overset{(M)}{\Longrightarrow} \underbrace{F(C) \subseteq F(F(C))}_{F(C) \in \mathcal{Y}}$$

$$\left(\mathcal{Y} = \{B \subseteq X;\ B \subseteq F(B)\}\right)$$

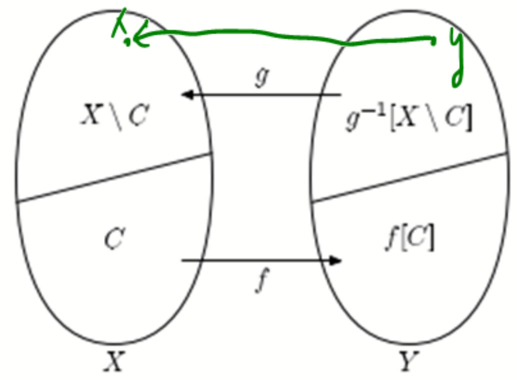
$$F(C) \subseteq F(F(C))$$

$$F(C) \subseteq C = \bigcup \underset{\substack{\uparrow \\ F(C) \in \mathcal{Y}}}{\mathcal{Y}}$$

$$\boxed{F(C) \subseteq C}$$

$$\Downarrow \quad \Downarrow$$

$$\boxed{F(C) = C}$$



$$g[Y \setminus f[C]] = X \setminus C$$

$$X \setminus g[Y \setminus f[C]] = C$$

$h : X \longrightarrow Y$

$$h(x) = \begin{cases} f(x); & x \in C \\ y; & g(y) = x \\ & x \notin C \end{cases}$$

① h je zobr. z X do Y

$x \in X$ — $x \in C$ ✓ $h(x) = f(x)$

$x \notin C$ ② $(\exists! y \in Y)\ g(y) = x$

Najviac 1 $y \in Y$ t. ž. $g(y) = x$.

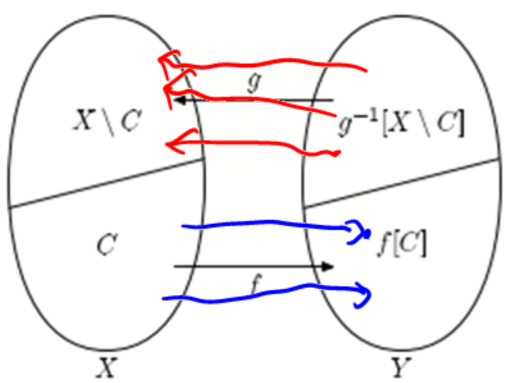
[Lebo g je inj.]

$x \in X \setminus C = g[Y \setminus f[C]]$

$\Leftrightarrow$ ex. $y \in Y$ t. ž. $g(y) = x$

② h je surj.

$$h(x) = \begin{cases} f(x), & \text{ak } x \in C, \\ y, & \text{kde } y \in Y \text{ je prvok s vlastnosťou } g(y) = x \text{ ak } x \notin C. \end{cases}$$



Ak $y \in Y$.

(A) $y \in f[C]$

Ex. $c \in C$ t. ž. $f(c) = y$.

$h(c) = f(c) = y$

(B) $y \notin f[C]$

(β) $y \notin f[C]$: $y \in Y \setminus f[C]$

$x = g(y)$ . . . $h(x) = y$ $\quad x \notin C$

$\hookrightarrow x \in g[Y \setminus f[C]] = X \setminus C$

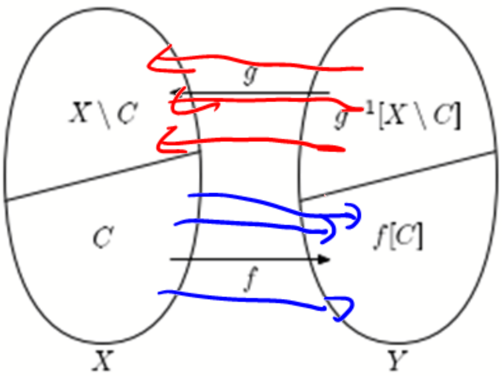
$$g[Y \setminus f[C]] = X \setminus C$$

$$X \setminus g[Y \setminus f[C]] = C$$

(3) h je <u>injekcia</u>

$$h(x) = \begin{cases} f(x), & \text{ak } x \in C, \\ y, & \text{kde } y \in Y \text{ je prvok s vlastnosťou } g(y) = x \text{ ak } x \notin C. \end{cases}$$

$h(x_1) = h(x_2) \Rightarrow x_1 = x_2$



(A) $x_{1,2} \in C$

$h(x_1) = h(x_2) \Rightarrow f(x_1) = f(x_2) \xRightarrow{f \text{ je inj.}} x_1 = x_2$

(B) $x_{1,2} \notin C$ $\quad h(x_1) = y = h(x_2)$

$$x_1 = g(y) = x_2$$

(C) $x_1 \in C$, $x_2 \notin C$ $\quad h(x_1) = h(x_2) = y$

NEMÔŽE NASTAŤ

$y = h(x_1) = f(x_1)$ $\quad g(y) = x_2$

$\boxed{y \in f[C]}$ $\quad x_2 \in X \setminus C = g[Y \setminus f[C]]$

SPOR

g je inj. $\quad \boxed{y \in Y \setminus f[C]}$

$$g[Y \setminus f[C]] = X \setminus C$$

$$X \setminus g[Y \setminus f[C]] = C$$

$\square$

① $A \subseteq B \Rightarrow f[A] \subseteq f[B]$

② $A \subseteq B \Rightarrow C \setminus A \supseteq C \setminus B$

③ $(\forall B \in \mathcal{Y})\ B \subseteq \bigcup \mathcal{Y}$

④ $(\forall B \in \mathcal{Y})\ B \subseteq D \Rightarrow \bigcup \mathcal{Y} \subseteq D$

⑤ $A \subseteq B \land B \subseteq C \Rightarrow A \subseteq C$