Binomické koeficienty

21. marca 2024

Definícia a základné vlastnosti

Definícia

Pre ľubovoľné $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$ definujme *binomický koeficient* $\binom{n}{k}$ ako počet všetkých k-prvkových podmnožín množiny n.

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

Pre reálne n

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$
$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

Binomická veta

$$(1+t)^0 = 1$$
$$(1+t)^1 = 1 + t$$
$$(1+t)^2 = 1 + 2t + t^2$$
$$(1+t)^3 = 1 + 3t + 3t^2 + t^3$$
$$(1+t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$$
$$(1+t)^5 = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$$
$$(1+t)^6 = 1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6$$

Binomická veta

Veta (Binomická veta)

Pre ľubovoľné $t \in \mathbb{C}$ *a* $n \in \mathbb{N}_0$ *platí*

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k. \tag{1}$$

Pre ľubovoľné $x, y \in \mathbb{C}$ *a* $n \in \mathbb{N}_0$ *platí*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{2}$$

Binomická veta

$$(1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

- počet výberov pri roznásobovaní;
- dôkaz matematickou indukciou

Binomická veta

$$
\begin{aligned}
(1+t)^{n+1} &= (1+t)(1+t)^n \\
&= (1+t)\sum_{k=0}^{n} \binom{n}{k} t^k \\
&= \sum_{k=0}^{n} \binom{n}{k} t^k + \sum_{k=1}^{n+1} \binom{n}{k-1} t^k \\
&= \sum_{k=0}^{n+1} \left(\binom{n}{k} + \binom{n}{k-1} \right) t^k \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} t^k
\end{aligned}
$$

Binomická veta

$$\begin{aligned}
(1+t)^{n+1} &= (1+t)(1+t)^n \\
&= (1+t)\sum_{k=0}^{n}\binom{n}{k}t^k \\
&= \sum_{k=0}^{n}\binom{n}{k}t^k + t\sum_{k=0}^{n}\binom{n}{k}t^k \\
&= \sum_{k=0}^{n}\binom{n}{k}t^k + \sum_{k=0}^{n}\binom{n}{k}t^{k+1} \\
&= \sum_{k=0}^{n}\binom{n}{k}t^k + \sum_{k=1}^{n+1}\binom{n}{k-1}t^k \\
&\overset{(*)}{=} \sum_{k=0}^{n+1}\binom{n}{k}t^k + \sum_{k=0}^{n+1}\binom{n}{k-1}t^k \\
&= \sum_{k=0}^{n+1}\left(\binom{n}{k}+\binom{n}{k-1}\right)t^k \\
&= \sum_{k=0}^{n+1}\binom{n+1}{k}t^k
\end{aligned}$$

Striedavé znamienka

Dôsledok

Pre ľubovoľné $n \in \mathbb{N}_0$ také, že $n \geq 1$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

A teda platia aj rovnosti

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} = 2^{n-1}.$$

Párne a nepárne

$$\sum_{0 \leq 2j \leq n} \binom{n}{2j} = \sum_{0 \leq 2j+1 \leq n} \binom{n}{2j+1}$$

$$\sum_{j=0}^{\infty} \binom{n}{2j} = \sum_{j=0}^{\infty} \binom{n}{2j+1} = 2^{n-1}$$

Kombinatorický argument, že máme rovnako veľa párnych a nepárnych podmnožín?

Párne a nepárne

$$
\begin{array}{rl}
(1+1)^n = & \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{2k} + \binom{n}{2k+1} + \dots \\
(1-1)^n = & \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{2k} - \binom{n}{2k+1} + \dots \\
\hline
2^n = & 2\binom{n}{0} + 2\binom{n}{2} + \cdots + 2\binom{n}{2k} + \dots
\end{array}
$$

Násobky trojky a štvorky

$$\sum_{k=0}^{\infty} \binom{n}{3k} = ?$$

$$\sum_{k=0}^{\infty} \binom{n}{4k} = ?$$

Derivácia

$$f(t) = (1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

$$f'(t) = n(1+t)^{n-1}$$

$$f'(t) = \sum_{k=1}^{n} k \binom{n}{k} t^{k-1}$$

$$\sum_{k=0}^{n} k \binom{n}{k} = n2^{n-1}$$

Iné odvodenie

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

$$\sum_{k=0}^{n} k\binom{n}{k} = \sum_{k=0}^{n}(n-k)\binom{n}{n-k}$$

Iné odvodenie

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$$

$$S = \sum_{k=0}^{n} k\binom{n}{k}$$

$$S = \sum_{k=0}^{n} (n-k)\binom{n}{k}$$

$$2S = n\sum_{k=0}^{n} \binom{n}{k} = n2^{n}$$

Integrovanie

$$f(t) = (1+t)^n = \sum_{k=0}^{n} \binom{n}{k} t^k$$

$$\int_0^x f(t)\,\mathrm{d}t = \frac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^{n} \binom{n}{k} \frac{x^{k+1}}{k+1}$$

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1}-1}{n+1}$$

Vlastnosti binomických koeficientov

$$k\binom{n}{k} = n\binom{n-1}{k-1} \tag{3}$$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \tag{4}$$

Vlastnosti binomických koeficientov

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

- Výber k-člennej komisie a predsedu.
- $\{(x, A); x \in A, A \subseteq M, |A| = k\}$

Vlastnosti binomických koeficientov

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

$$\binom{n}{k}\binom{k}{l} = \binom{n}{l}\binom{n-l}{k-l}$$

Vlastnosti binomických koeficientov

$$\binom{n}{k} < \binom{n}{k+1} \qquad \text{ak } 2k+1 < n$$

$$\binom{n}{k} > \binom{n}{k+1} \qquad \text{ak } 2k+1 > n$$

Vandermondova identita

Tvrdenie

Pre ľubovoľné $m, n \in \mathbb{N}_0$ *platí*

$$\sum_{j=0}^{s} \binom{m}{j} \binom{n}{s-j} = \binom{m+n}{s}. \tag{5}$$

Zovšeobecnenie binomického koeficientu

Definícia

Pre $t \in \mathbb{C}$ a $k \in \mathbb{N}_0$ označme

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}$$

$$\binom{t}{0} = 1$$

Zovšeobecnenie binomickej vety

Veta

Nech $t \in \mathbb{R}$. *Nech* $x \in \mathbb{R}$, $|x| < 1$. *Potom platí rovnosť*

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k. \tag{6}$$

(T.j. rad na pravej strany hodnosti konverguje k hodnote uvedenej na ľavej strane.)

Zovšeobecnenie binomickej vety

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k$$

$$(1+x)^t = 1 + tx + \frac{t(t-1)}{2!}x^2 + \frac{t(t-1)(t-2)}{3!}x^3 + \cdots +$$
$$+ \frac{t(t-1)(t-2)\cdots(t-k+1)}{k!}x^k + \ldots$$

Taylorov rad funkcie $f(x) = (1+x)^t$ v nule.

Taylorov rad pre $f(x) = (1+x)^t$

$$f(x) = (1+x)^t$$

$$f^{(k)}(x) = t(t-1)\cdots(t-k+1)x^{t-k}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \ldots$$

$$(1+x)^t = 1 + tx + \frac{t(t-1)}{2}x^2 + \cdots + \binom{t}{k}x^k + \ldots$$

$$(1+x)^t = \sum_{k=0}^{\infty}\binom{t}{k}x^k$$

$\binom{n}{k}$ pre záporné n

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

$$\binom{-n}{k} = \frac{(-n)(-n-1)\cdots(-n-k+2)(-n-k+1)}{k!}$$

$$\binom{n+k-1}{k} = \frac{(n+k-1)(n+k-2)\cdots(n+1)n}{k!}$$

Prípad $n = -1$

$$\binom{-1}{k} = (-1)^k$$

$$(1+x)^{-1} = \sum_{k=0}^{\infty} (-1)^k x^k$$
$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Záporný exponent

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

Záporný exponent

$$\frac{1}{(1+x)^n} = (1 + x + x^2 + x^3 + \dots)^n.$$

$$a_1 + a_2 + \dots + a_n = k.$$

$$\binom{n+k-1}{k}$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

Multinomická veta

$$(x_1 + x_2 + \cdots + x_k)^n = ?$$

$$\underbrace{(x_1 + x_2 + \cdots + x_k)(x_1 + x_2 + \cdots + x_k)\cdots(x_1 + x_2 + \cdots + x_k)}_{n\text{-krát}}$$

Koľkými spôsobmi môžeme dostať $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$?

$$\binom{n}{a_1, a_2, \ldots, a_k} = \frac{n!}{a_1! a_2! \cdots a_k!}$$

Multinomická veta

Veta (Multinomická veta)

Pre ľubovoľné $x_1, x_2, \ldots, x_n \in \mathbb{R}$ *platí rovnosť*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\substack{a_1+\cdots+a_k=n \\ a_i \geq 0}} \binom{n}{a_1, a_2, \ldots, a_k} x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$$