# Topological spaces

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# Definition of topology

**Definition**

Let $X$ be a set. A system $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on the set $X$ if:

(O1). $\emptyset, X \in \mathcal{T}$.

(O2). If $A, B \in \mathcal{T}$, then also $A \cap B \in \mathcal{T}$.

(O3). if $A_i \in \mathcal{T}$ for each $i \in I$, then als $\bigcup_{i \in I} A_i \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on $X$, then the pair $(X, \mathcal{T})$ is called a *topological space.* The sets belonging to $\mathcal{T}$ are called *open sets* in the space $(X, \mathcal{T})$.

# Definition of topology

- There are various ways to describe a topology – this axiomatization is relatively simple.
- One possible viewpoint is that we are "measuring" distance – but using bigger and smaller neighborhood rather than using numbers.

# Examples of topologies

- $\mathcal{T}_{ind} = \{\emptyset, X\}$ = *indiscrete topology*
- $\mathcal{T}_{disc} = \mathcal{P}(X)$ = *discrete topology*
- Sierpiński space

# Topology given by a metric

$$B(a,r)=\{x\in X; d(x,a)<r\}$$

- A point $a \in U$ is called an *interior point* of a set $U$ if there is a real number $r>0$ such that $B(a,r)\subseteq U$.
- If each point of a set $U$ is its interior point we say that $U$ is *open in the metric space* $(X,d)$.

$$(X,\mathcal{T}_d)$$

# Closed sets

## Definition

Let $X$ be a topological space and $C \subseteq X$.
A subset $C$ is called a *closed set*, it the complement $X \setminus C$ is an open set.
If $C$ is both closed and open in $X$, we say that it is a *clopen set*.

*Sets are not doors. They can be open, closed, both, or neither.*
Neznámy autor

# Closed sets

## Proposition

*Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C}$ be the system of all closed sets in $X$. Then the following holds:*

(C1). $\emptyset, X \in \mathcal{C}$.

(C2). *Ak* $A, B \in \mathcal{C}$ *tak,* $A \cup B \in \mathcal{C}$.

(C3). *Ak* $A_i \in \mathcal{C}$ *pre všetky* $i \in I$ *(pričom* $I \neq \emptyset$*), tak aj* $\bigcap_{i \in I} A_i \in \mathcal{C}$.

# Closed sets

## Proposition

*Let $X$ be an arbitrary set. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a system of sets satisfying the conditions* (C1), (C2), (C3). *Then*

$$\mathcal{T} = \{X \setminus C; C \in \mathcal{C}\}$$

*is a topology on $X$.*
*Moreover, the closed sets in $(X, \mathcal{T})$ are precisely the sets belonging to $\mathcal{C}$.*

# Cofinite and cocountable topology

$$\mathcal{T}_{cof} = \{\emptyset\} \cup \{X \setminus F; F \text{ is a finite subset of } X\}$$

$$\mathcal{T}_{coc} = \{\emptyset\} \cup \{X \setminus F; F \text{ is a countable subset of } X\}$$

# Bases

## Definition

Let $(X, \mathcal{T})$ be a topological space. A system $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis for the topology* $\mathcal{T}$ if every open set $U$ is a union of some system of sets from $\mathcal{B}$.

$$(\forall U \in \mathcal{T})(\exists \mathcal{S} \subseteq \mathcal{B}) U = \bigcup \mathcal{S}$$

- $\mathcal{B} \subseteq \mathcal{T}$, i.e., every basic set is open
- Equivalent condition: For any $x \in X$ and any open neighborhood $U \ni x$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$$(\forall x \in U \in \mathcal{T})(\exists B \in \mathcal{B}) x \in B \subseteq U$$

# Bases

## Theorem

*If $\mathcal{B}$ is a basis for some topology on $X$ then:*

(B1). *$\mathcal{B}$ covers $X$, i.e.,*

$$\bigcup \mathcal{B} = X.$$

(B2). *If $B_{1,2} \in \mathcal{B}$ both contain a point $x \in X$ then there exists $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.*

$$(\forall x \in X)[x \in B_{1,2} \in \mathcal{B} \Rightarrow (\exists B \in \mathcal{B}) x \in B \subseteq B_1 \cap B_2]$$

# Bases

Conversely, if $\mathcal{B} \subseteq \mathcal{P}(X)$ stafisfies $(\mathrm{B1})$ and $(\mathrm{B2})$ then the set of all unions of subsystems of $\mathcal{B}$ gives a topology $\mathcal{T}$ on $X$.

$$\mathcal{T} = \{\bigcup \mathcal{C}; \mathcal{C} \subseteq \mathcal{B}\}$$

Moreover, $\mathcal{B}$ is a basis for $\mathcal{T}$.

# Examples of bases

- $\mathcal{B}_{disc} = \{\{x\}; x \in X\}$
- $\mathcal{B}_{ind} = \{X\}$

# Topology given by a metric

$B(a, r) = \{x \in X; d(x, a) < r\}$

$$\mathcal{B} = \{B(a, r); a \in X, r \in \mathbb{R}, r > 0\}$$

Notation: $\mathcal{T}_d$
Another basis: $\mathcal{B}' = \{B(a, r); a \in X, r \in \mathbb{Q}, r > 0\}$

# Sorgenfrey line

Sorgenfrey line (Lower limit topology)
$X = \mathbb{R}$ s topológiou danou bázou

$$\mathcal{B} = \{\langle a, b); a, b \in \mathbb{R}, a < b\}.$$

Notation: $\mathbb{R}_l$ or $\mathcal{T}_l$.

- $\langle a, b)$ is clopen
- $\mathcal{T}_e \subseteq \mathcal{T}_l$

# Subbasis

## Definition

Nech $(X, \mathcal{T})$ je topológia a $\mathcal{S} \subseteq \mathcal{T}$. Hovoríme, že $\mathcal{S}$ *subbáza* topológie $\mathcal{T}$, ak

$$\mathcal{B} = \{\bigcap \mathcal{F}; \mathcal{F} \subseteq \mathcal{S}, \mathcal{F} \text{ je neprázdna konečná množina}\}$$

tvorí bázu topológie $\mathcal{T}$.

T.j. $\mathcal{S}$ je subbáza práve vtedy, keď prieniky konečne veľa množín z $\mathcal{S}$ tvoria bázu.

# Subbasis

**Theorem**

*If $\mathcal{S}$ is a subbasis for some topology on $X$ then:*

(S1). *$\mathcal{S}$ covers $X$, i.e.,*

$$\bigcup \mathcal{S} = X.$$

*Conversely, if $\mathcal{S} \subseteq \mathcal{P}(X)$ fulfills (S1) then the set of finite intersections of finite subsystems of $\mathcal{S}$, i.e.,*

$$\mathcal{B} = \{\bigcap \mathcal{F}; \mathcal{F} \subseteq \mathcal{S}, \mathcal{F} \text{ is a non-empty finite set}\}$$

*is a basis for a topology on $X$.*

# Neighborhood

## Definition

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. a subset $N \subseteq X$ is called a *neighborhood of the point* $x$ if there exists an open set $U$ with $x \in U \subseteq N$. If $N$ is moreover an open set, we say that it is an *open neighborhood* of $x$.
We will denote by $\mathcal{N}_x$ the system of all neighborhoods of $x$, and by $\mathcal{O}_x$ the system of all open neighborhoods of $x$.

# Neighborhood basis

**Definition**

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Let $\mathcal{B}_x \subseteq \mathcal{N}_x$, i.e., $\mathcal{B}_x$ is a system consisting of some neighborhoods of $x$. We say that $\mathcal{B}_x$ is a *neighborhood basis at the point* $x$ if, for any open set $U$ containing $x$, there exists $B \in \mathcal{B}_x$ with $B \subseteq U$.

$$(\forall U \in \mathcal{O}_x)(\exists B \in \mathcal{B}_x) x \in B \subseteq U$$

- Open balls are a neighborhood basis for $\mathcal{T}_d$.
- Closed intervals in $\mathbb{R}$ (closed balls in $\mathbb{R}^n$).

# Neighborhood basis

## Theorem

*Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{T}$ (i.e., $\mathcal{B}$ is a system of open sets). Then $\mathcal{B}$ is a base for $X$ if and only if $\mathcal{B}_x = \{B \in \mathcal{B}; x \in B\}$ is a neighborhood basis at $x$ for each $x \in X$.*

# Sorgenfrey line

## Example

$$\mathcal{B} = \{\langle a, b); a, b \in \mathbb{R}, a < b\}$$

$$\begin{aligned}
\mathcal{B}_x &= \{\langle a, b); a, b \in \mathbb{R}; a \leq x < b\} \\
\mathcal{B}'_x &= \{\langle x, b); b \in \mathbb{R}; x < b\} \\
\mathcal{B}''_x &= \{\langle x, b); b \in \mathbb{Q}; x < b\}
\end{aligned}$$

If $x$ is irrational, the system

$$\mathcal{C}_x = \{\langle a, b); a, b \in \mathbb{Q}; a \leq x < b\}$$

is not a neighborhood basis at $x$.

# Neighborhood basis

## Theorem

*Let $(X, \mathcal{T})$ be a topological spaces.*
*For every $x \in X$, let $\mathcal{B}_x$ be a neighborhood basis at $x \in X$ consisting only of open sets, i.e., $\mathcal{B}_x \subseteq \mathcal{T}$. Then the following holds:*

(BO1). *For each $B \in \mathcal{B}_x$ we have $x \in B$.*

(BO2). *If $U_{1,2} \in \mathcal{B}_x$ then there exists $U \in \mathcal{B}_x$ such that $U \subseteq U_1 \cap U_2$.*

(BO3). *if $y \in U \in \mathcal{B}_x$ then there exists $V \in \mathcal{B}_y$ such that $V \subseteq U$.*

# Neighborhood basis

**Theorem**

*Conversely, suppose that for every point $x \in X$ we have a system $\mathcal{B}_x \subseteq \mathcal{P}(X)$ and that these systems fulfill the conditions* (BO1)–(BO3). *Then*

$$\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$$

*fulfills the conditions* (B1) *and* (B2) *and thus it is a basis for a topology $\mathcal{T}$ on the set $X$. Moreover, $\mathcal{B}_x$ is a neighborhood basis at $x$ for this topology $\mathcal{T}$.*

# Moore plane

## Example

On $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2; x_2 \geq 0\}$ at the points $b = (b_1, b_2)$ with $b_2 > 0$ we take the basis given by the Euclidean metric, i.e., $B(b, r) = \{(x_1, x_2) \in \Gamma; \|x - y\|_2 = \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2} < r\}$, and for the points where the second coordinate is zero"

$$\mathcal{B}_{(b_1,0)} = \{(b_1, 0)\} \cup \{\sqrt{(x_1 - b_1)^2 + (x_2 - r)^2} < r\}$$

# Moore plane

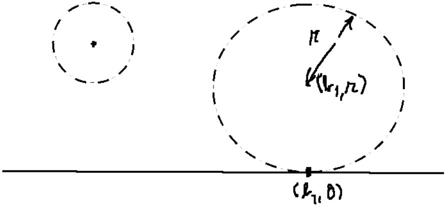

Figure: Bázové množiny v Mooreovej rovine.

# Closure

## Definition

Let $(X, \mathcal{T})$ be topological space and $A \subseteq X$. Then the set

$$\overline{A} = \bigcap \{C; A \subseteq C \subseteq X; C \text{ je uzavretá podmnožina } X\} \qquad (1)$$

is called the *closure of the set A.*

Sometimes we use the notation $\mathrm{cl}(A)$ or $\mathrm{cl}_{\mathcal{T}}(A)$ – for example, if we want to work with closures of the same set in two different topologies.

# Closure

## Proposition

*Nech $(X, \mathcal{T})$ je topologický priestor.*

(i) *The set $\overline{A}$ is closed for any $A \subseteq X$ .*

(ii) *A set $A \subseteq X$ is closed iff $A = \overline{A}$.*

(iii) *If $A \subseteq C \subseteq X$ and $C$ is a closed set then $\overline{A} \subseteq C$.*

## Proposition

*Let $(X, \mathcal{T})$ be a topological space and $A$, $B$ be subsets of $X$. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.*

$$A \subseteq B \quad \Rightarrow \quad \overline{A} \subseteq \overline{B} \tag{2}$$

# Closure

## Theorem

*For the closure of sets in a topological space $X$ we have:*

(CL1). $\overline{\emptyset} = \emptyset$;

(CL2). $A \subseteq \overline{A}$;

(CL3). $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(CL4). $\overline{(\overline{A})} = \overline{A}$

*Conversely, if an operator* $\overline{\phantom{A}} \colon \mathcal{P}(X) \to \mathcal{P}(X)$ *fulfills the conditions* (CL1)–(CL4) *and if we define*

$$\mathcal{C} = \{A \subseteq X; \overline{A} = A\},$$

*the the system* $\mathcal{C}$ *fulfills* (C1)–(C3). *(Thus the sets such that* $A = \overline{A}$ *are exactly the closed set of the topology* $\mathcal{T}$ *obtained by taking the complements of the sets from* $\mathcal{C}$.*)*

# Closure

- Description "from above" and "from below"
- In metric spaces: Closure = limits of sequences.
- In topological spaces: Closure = limits of nets.

# Closure

### Proposition

*Let $(X, \mathcal{T})$ be a topological space, $x \in X$, $A \subseteq X$.*
*Then $x \in \overline{A}$ if and only if every neighborhood $U$ of the point $x$ intersects $A$.*

$$x \in \overline{A} \qquad \Leftrightarrow \qquad (\forall U \in \mathcal{N}_x)(A \cap U \neq \emptyset)$$

# Locally finite systems

## Definition

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{S}$ by a system of subsets of $X$. The system $\mathcal{S}$ is called *locally finite* if, for every point $x \in X$, there exists a neighborhood $U \ni x$ which intersects only finitely many sets from $\mathcal{S}$. (I.e., the set $\{S \in \mathcal{S}; S \cap U \neq \emptyset\}$ is finite.)

# Locally finite systems

**Theorem**

*Let $\{A_i; i \in I\}$ be a locally finite system*

$$\overline{\bigcup_{i\in I} A_i} = \bigcup_{i\in I} \overline{A_i}.$$

**Corollary**

*Union of a locally finite system of closed sets if a closed set.*

# Interior of a set

## Definition

Nech $(X, \mathcal{T})$ je topologický priestor a $A \subseteq X$. Potom *vnútro množiny A* definujeme ako

$$\operatorname{Int} A = \bigcup \{U \subseteq A;\, U \text{ je otvorená v } X\}.$$

## Proposition

*Nech $(X, \mathcal{T})$ je topologický priestor, $A$ je podmnožina $X$. Potom:*

$$\overline{A} = X \setminus \operatorname{Int}(X \setminus A)$$
$$\operatorname{Int} A = X \setminus \overline{X \setminus A}$$

# Interior of a set

## Proposition

*Let* $(X, \mathcal{T})$ *be a topological space and* $A, B \subseteq X$*. Then we have:*

(i) *If* $A \subseteq B$ *then* $\operatorname{Int} A \subseteq \operatorname{Int} B$.

(ii) $\operatorname{Int} A \subseteq A$

(iii) $\operatorname{Int}(A \cap B) = \operatorname{Int} A \cap \operatorname{Int} B$

(iv) *The set* $\operatorname{Int} A$ *is open. For any open set* $U \subseteq X$ *we have* $\operatorname{Int} U = U$.

(v) $\operatorname{Int}(\operatorname{Int} A) = \operatorname{Int} A$

# Dense sets

### Definition

Let $(X, \mathcal{T})$ be a topological space. A subset $D \subseteq X$ is *dense* in $X$ if $\overline{D} = X$, i.e., the closure of $D$ is the whole space.

An equivalent characterization is that $D$ intersects every non-empty open set.

$$(\forall U \in \mathcal{T} \setminus \{\emptyset\}) D \cap U \neq \emptyset$$

We can use elements from some basis $\mathcal{B}$ instead the whole topology $\mathcal{T}$ in this condition.

# Dense sets

## Proposition

*Let $(X, \mathcal{T})$ be a topological space, $U, D \subseteq X$. If $D$ is a dense set and $U$ is an open set then*

$$\overline{U \cap D} = \overline{U}.$$