Separable spaces and countability axioms

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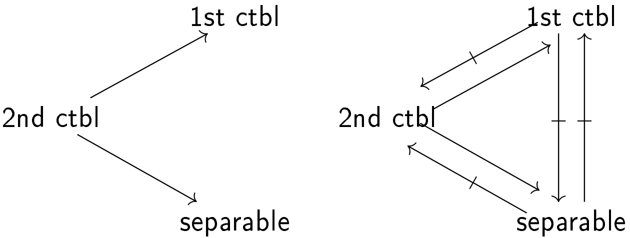
Separability and countability axioms



First countable	A countable neighborhood base at each point
Second countable	A countable base for the topology
Separable	Countable dense subset

First countable spaces

Definition

Let $(X, \mathcal{T})$ be a topological space. The space X is first countable, if every point $x \in X$ has a countable neighborhood base $\mathcal{B}_x$.

- Metric spaces: $\mathcal{B}_x = \{B(x,r); r \in \mathbb{Q}\}$
- Sorgenfrey line $\mathbb{R}_l$

Second countable spaces

Definition

A topological space is *second countable* if there exists a countable base $\mathcal{B}$ for X

Example: $(\mathbb{R}, \mathcal{T}_e)$

Second countable spaces

Proposition

Every first countable space is second countable

- $(X, \mathcal{T}_{disc})$ is second countable $\Leftrightarrow |X| \leq \aleph_0$.
- Sorgenfrey line $\mathbb{R}_l = (\mathbb{R}, \mathcal{T}_l)$ is not second countable.

Separable spaces

Definition

A topological space $(X, \mathcal{T})$ is *separable*, if there exists a countable subset $A \subseteq X$ such that $\overline{A} = X$.

- The discrete space $(X, \mathcal{T}_{disc})$ is separable $\Leftrightarrow$ X is a countable set.
- The real line with the usual topology $(\mathbb{R}, \mathcal{T}_e)$

Separable spaces

Proposition

Every second countable space is separable.

The converse implication is not true: Sorgenfrey line, Moore plane

Separable metrizable spaces

Proposition

If X is a separable metrizable space, then X is second countable. In the other words, for a metrizable space X we have: X is separable iff X is second countable.

Separable spaces

- The spaces ℓ_p, $1 < p < \infty$ are separable.
- The spaces c and c_0 with the sup-norm are separable.
- Priestor ℓ_∞ with the sup-norm is not separable.