Continuity

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Continuity at a point

Metric spaces: $f\colon (X,d) \to (Y,d')$

$$(\forall \varepsilon > 0)(\exists \delta > 0) d(x,a) < \delta \Rightarrow d'(f(x), f(a)) < \varepsilon.$$

Continuity at a point

Definition

Let $f\colon X \to Y$ be a function, X and Y be topological spaces. Let $a \in X$. The function f is *continuous at the point a* if, for any open neighborhood V of the point $f(a)$, there exists an open neighborhood U of the point a such that $f[U] \subseteq V$.

$$(\forall V \in \mathcal{O}_{f(a)})(\exists U \in \mathcal{O}_a) f[U] \subseteq V \tag{1}$$

Continuity at a point

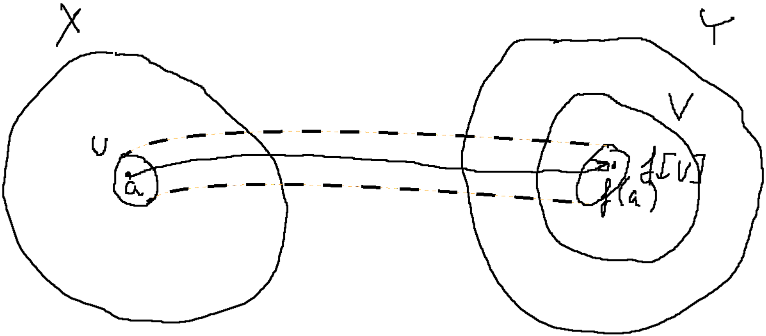

Figure: Definition of continuity at a point

Continuity at a point

It suffices to use a neighborhood base:

$$(\forall V \in \mathcal{B}_{f(a)})(\exists U \in \mathcal{B}_a) f[U] \subseteq V$$

Preimages of open sets

Definition

Let X, Y be topological spaces. A function $f\colon X \to Y$ is *continuous*, if it is continuous at every point $a \in X$.

Theorem

Let X, Y be topological spaces and $f\colon X \to Y$ be a function. The function f is continuous if for every open subset U of the space Y the preimage $f^{-1}[U]$ is open in X.

$$(\forall U \in \mathcal{T}_Y) f^{-1}[U] \in \mathcal{T}_X$$

Examples of continuous functions

- $f : (X, \mathcal{T}_{disc}) \to (Y, \mathcal{T}_Y)$
- $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_{ind})$
- constant function
- $id_X : (X, \mathcal{T}_1) \to (\mathcal{T}_2) \Leftrightarrow \mathcal{T}_2 \subseteq \mathcal{T}_1$

Preimages of basic sets

Proposition

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces, let $\mathcal{B}$ be a base for $\mathcal{T}_Y$, let $\mathcal{S}$ be a subbase $\mathcal{T}_Y$. Let $f : X \to Y$. The following conditions are equivalent:

(i) *f is continuous.*

(ii) *For each $V \in \mathcal{B}$ the preimage $f^{-1}[V]$ is an open set.*

(iii) *For each $W \in \mathcal{S}$ the preimage $f^{-1}[W]$ is an open set.*

Example

$f^{-1}[(-\infty, b)]$ and $f^{-1}[(a, \infty)]$ for $a, b \in \mathbb{R}$.

Composition of continuous functions

Theorem

Let $f: X \to Y$, $g: Y \to Z$ be continuous functions between topological spaces. Then the composition $g \circ f: X \to Z$ is continuous, too.

$$g \circ f^{-1}[U] = f^{-1}[g^{-1}[U]]$$

Image of the closure

Proposition

Let $f: X \to Y$ be a function for a topological space X to the topological space Y. The following are equivalent:

(i) *The function f is continuous.*

(ii) *For every closed subset C of the space Y, the preimage $f^{-1}[C]$ is a closed subset of X.*

(iii) *For every $A \subseteq X$ we have $f[\overline{A}] \subseteq \overline{f[A]}$.*

Image of the closure

$$C = \overline{f[A]}$$

$$f^{-1}[C] = f^{-1}[\overline{f[A]}]$$

$$A \subseteq f^{-1}[f[A]] \subseteq f^{-1}[\overline{f[A]}]$$

$$\overline{A} \subseteq f^{-1}[\overline{f[A]}]$$

$$f[\overline{A}] \subseteq \overline{f[A]}$$

Image of the closure

$$A = f^{-1}[C]$$

$$f[\overline{A}] \subseteq \overline{f[A]} = \overline{f[f^{-1}[C]]} \subseteq \overline{C} = C$$

$$f^{-1}[C] \subseteq \overline{f^{-1}[C]} \subseteq f^{-1}[C]$$

$$f^{-1}[C] = \overline{f^{-1}[C]}$$

Image of a dense set

Corollary

Let X, Y be topological spaces and $f\colon X \to Y$ be a continuous surjective function. If D is a dense subset of X then $f[D]$ is a dense subset of Y.

Definition

We say that Y is a *continuous image* of a space X if there exists a continuous surjective function $f\colon X \to Y$.

Corollary

Continuous image of a separable space is separable.