Homeomorphisms

October 14, 2024

Definition of a homemorphism

Definition

A map $h\colon X \to Y$ between topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ is called a *homeomorphism*, if h is bijective, continuous and the inverse f^{-1} is continuous.

$$U \in \mathcal{T}_X \Leftrightarrow h[U] \in \mathcal{T}_Y$$
$$V \in \mathcal{T}_Y \Leftrightarrow h^{-1}[V] \in \mathcal{T}_X$$

Homeomorphic spaces

Definition

Topological spaces $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ are *homeomorphic* if there exists a homeomorphism $h\colon X \to Y$.

Notation: $(X, \mathcal{T}_X) \cong (Y, \mathcal{T}_Y)$ or more briefly $X \cong Y$.

- $X \cong X$;
- $X \cong Y \Rightarrow Y \cong X$
- $X \cong Y \wedge Y \cong Z \Rightarrow X \cong Z$

Topological properties

Definition

A property P of topological spaces is called a *topological property* if, for any two homeomorphic spaces X and Y, the space X has the property P if and only iff Y has the property P.

I.e., topological properties are the properties which are preserved by homeomorphisms.

Intervals

- $\langle a, b \rangle \cong \langle c, d \rangle$
- $(a, b) \cong (c, d)$
- $\langle 0, 1 \rangle \ncong (0, 1)$ resp. $\langle 0, 1 \rangle \ncong \mathbb{R}$

Intervals and $\mathbb{R}$

$$(0,1) \cong (a,b) \cong (a,\infty) \cong (-\infty,b) \cong \mathbb{R}$$

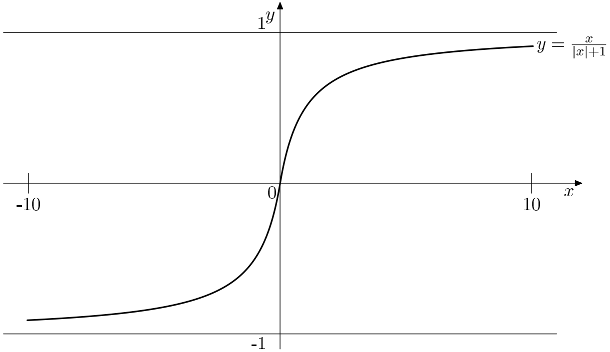


Figure: Example of a homeomorphism between $\mathbb{R}$ and $(-1,1)$

Interval and circle

$$S = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$$

$$(0, 1) \cong S \setminus \{x_0\}$$

$$h(t) = (\cos t, \sin t)$$

$$d(h(t_1), h(t_2)) = \left|2 \sin \frac{t_1 - t_2}{2}\right|$$

Interval and circle

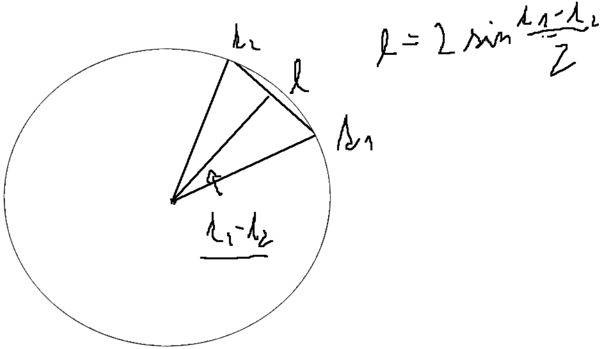
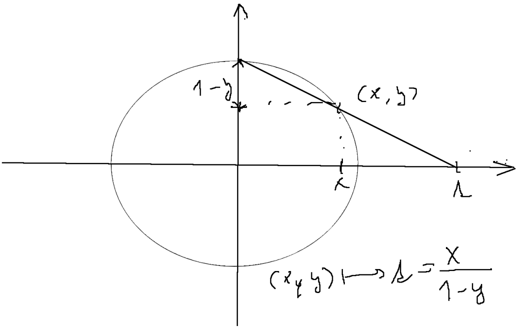


Figure: Distance of two points on a circle.

Stereografická projekcia

$S \setminus \{x_0\} \cong \mathbb{R}$



$$(x, y) \mapsto \frac{x}{1 - y}$$

Stereographic projection

$$S \setminus \{(0,1)\} \to \mathbb{R}$$

$$(x,y) \mapsto \frac{x}{1-y}$$

$$t \mapsto \left(\frac{2t}{t^2+1}, \frac{t^2-1}{t^2+1}\right)$$

Stereographic projection

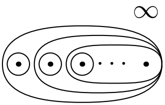
$$(x, y) \mapsto \frac{x}{1-y}$$

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

$$(x_1, x_2, \ldots, x_n) \mapsto \left(\frac{x_1}{1-x_n}, \frac{x_2}{1-x_n}, \ldots, \frac{x_{n-1}}{1-x_n}\right)$$

The space $C(\omega)$

- $X = \{0, 1, 2, \dots\} \cup \{\infty\}$
- $\mathcal{B}_x = \{\{x\}\}$ for $x \neq \infty$
- $\mathcal{B}_\infty =$ complements of finite subsets



$C(\omega)$

$$C(\omega) \cong \{0\} \cup \{\frac{1}{n}; n \in \mathbb{N} \setminus \{0\}\}$$

Closed and open maps

Definition

Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and let $f\colon X \to Y$ be a map.

The map f is *open* if, for any open set $U \in \mathcal{T}_X$, the image $f[U]$ otvorená is an open subset of the space Y.

The map f is *closed* if for any closed subset C of the space X, the image $f[C]$ is closed in Y.

For a map to be open, it is sufficient of images of basic sets are open.

Closed and open maps

Proposition

Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a bijection. The following conditions are equivalent:

a) The map f is open.

b) The map f is closed.

c) The inverse map f^{-1} is continuous.

Closed and open maps

Corollary

Let $f : X \to Y$ be a map between two topological spaces. Then:
a) The map f is a homeomorphism if and only if f is bijective, continuous and open.
b) The map f is a homeomorphism if and only if f is bijective, continuous and closed.