Subspaces

October 16, 2024

Topological constructions

- subspace
- quotient space
- topological sum
- product space
- Generalization: Initial and final topology.

Definition of a subspace

Definition

Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. If we define

$$\mathcal{T}_S = \{U \cap S; U \in \mathcal{T}\},$$

then $\mathcal{T}_S$ is a topology on S.
The pair $(S, \mathcal{T}_S)$ is called a *subspace* of the topological space X.
The topology $\mathcal{T}_S$ is called the *relative topology*
If S is an open subset of X, we say that it is an *open subspace*.
Similarly, S is called a *closed subspace* if S is a closed subset.

Base, closure, interior

- base $\mathcal{B}_S = \{U \cap S; U \in \mathcal{B}\}$
- neighborhood base $\mathcal{B}'_x = \{B \cap S; B \in \mathcal{B}_x\}$
- closedness: $C = C' \cap X$.
- $\mathrm{cl}_S(A) = \mathrm{cl}_X(A) \cap S$
- $\mathrm{Int}_S(A) = \mathrm{Int}_X(A) \cap S$

Subspace

Example

- subspace of a discrete space
- subspace of an indiscrete space
- subspace of a metric space

$$B_S(x,r) = \{y \in S; d(x,y) < r\} = B(x,r) \cap S.$$

Subspace

Proposition

If S is a subspace of T and T is a subspace of X then S is a subspace of X.
If S, T are subspaces of X and $S \subseteq T$ then also S is a subpace of T.

Subspace

Proposition

Let $f: X \to Y$ be a function. Let S be a subspace of the space X and let T be a subspace of Y. Moreover, let as assume that $f[S] \subseteq T$. Then:
a) If $f: X \to Y$ is continuous then the restriction $f|_S: S \to Y$ is continuous, too.
b) The function $f: X \to Y$ is continuous if and only if $f: X \to T$ is continuous.

Definition of an embedding

Definition

A map $i\colon S \to X$ between the topological spaces S and X is called an *embedding* if $i\colon S \to i[S]$ is a homeomorphism between S and the subspace $i[S]$ of the space X. An embedding is denoted as $i\colon S \hookrightarrow X$.

Embedding

Proposition

Let $S \subseteq X$, where $(S, \mathcal{T}_S)$ and $(X, \mathcal{T}_X)$ are topological spaces. Let us define $i \colon S \to X$ by $i(x) = x$ for any $x \in S$. Then we have: $(S, \mathcal{T}_S)$ is a subspace of $(X, \mathcal{T})$ if and only if $i \colon (S, \mathcal{T}_S) \hookrightarrow (X, \mathcal{T}_X)$ is an embedding.

Embedding

Proposition

If $f\colon X \hookrightarrow Y$ *and* $g\colon Y \hookrightarrow Z$ *are embeddings, then the composition* $g \circ f\colon X \to Z$ *is an embedding, too.*

Corollary

If $f\colon X \hookrightarrow Y$ *is an embedding and* S *is a subspace of* X *then the restriction* $f|_S\colon S \to Y$ *as well.*

Embedding

Proposition

Let X be a topological space and S be a subspace of S. Let $i_S \colon S \hookrightarrow X$ be the embedding of S into X.
Let Y be a topological space and $g \colon Y \to S$ je zobrazenie. Potom g je spojité práve vtedy, keď $e \circ g$ je spojité.

Hereditary properties

Definition

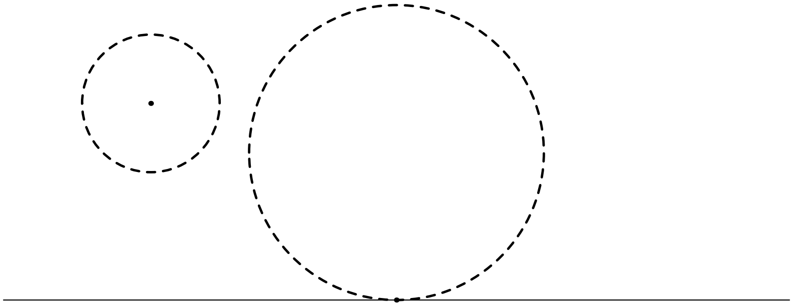
A topological property P (or a class of topological spaces) is called *hereditary* if, for any space with the property P, every its subspace has the property P, too.

We use the name *open (closed) hereditary* property if an analogous claim holds for open (closed) subspaces.

Countability axioms

- Any subspace of a first countable space is first countable.
- Any subspace of a second countable space is second countable.
- Every subspace of a separable space is separable. (Moore plane is an example of a space which is not hereditarily separable.)

Moore plane



Cover

Definition

Let X be a topological space. A family $\mathcal{C} = \{C_i; i \in I\}$ of subsets of the set X is called a *cover* of the space X if

$$\bigcup \mathcal{S} = \bigcup_{i \in I} A_i = X.$$

If every element of the cover $\mathcal{C}$ is an open set then $\mathcal{C}$ is an *open cover.*
If every element of $\mathcal{C}$ is a closed set then $\mathcal{C}$ is a *closed cover.*
If $\mathcal{C}$ is a locally finite system then $\mathcal{C}$ is a *locally finite cover.*

Open cover

Proposition

Let $\{U_i; i \in I\}$ be an open cover of a topological space X. Let $f\colon X \to Y$ be a map into a topological space Y.
If the restriction $f|_{U_i}\colon U_i \to Y$ is continuous for every $i \in I$ then the map f is also continuous.

$$f^{-1}[V] = \bigcup_{i\in I}(f^{-1}[V] \cap U_i) = \bigcup_{i\in I}(f|_{U_i})^{-1}[V]$$

Locally finite closed cover

Proposition

Let X, Y be topological spaces and $f\colon X \to Y$ be a map. Let $\{C_i; i \in I\}$ be a locally finite closed cover of X.
If the restriction $f|_{C_i}\colon C_i \to Y$ is continuous for every $i \in I$ then the map f is also continuous.

$$f^{-1}[C] = \bigcup_{i\in I}(f^{-1}[C] \cap C_i) = \bigcup_{i\in I}(f|_{C_i})^{-1}[C]$$