Quotient maps

October 16, 2024

Equivalence relations and surjections

- A surjection $f\colon X \to Y$ induces the equivalence relation

$$x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2),$$

  the equivalence classes are $f^{-1}(y)$.
- If $\sim$ is an equivalence relation the for the quotient set $X/\sim = \{[x]; x \in X\}$ we have a surjection

$$p\colon X \to X/\sim$$
$$p\colon x \mapsto [x]$$

- Hence any equivalence relation can be determined by a surjective function and, conversely, every equivalence relation gives a canonical surjective function.

Topology from an equivalence relation

Proposition

Let $(X, \mathcal{T})$ be a topological space and $\sim$ be an equivalence relation on the set Y. Let us denote $p\colon X \to X/\sim$ the map given by $p(x) = [x]$. Let us define

$$\mathcal{T}_\sim = \{V \subseteq X/\sim; p^{-1}[V] \in \mathcal{T}\}.$$

Then $\mathcal{T}_\sim$ is a topology on the set X.
Topologický priestor $(X/\sim, \mathcal{T}_\sim)$ budeme nazývať faktorový priestor priestoru X podľa relácie $\sim$.

$p\colon X \to X/\sim$ je surjekcia a platí:

$$V \in \mathcal{T}_\sim \qquad \Leftrightarrow \qquad p^{-1}[V] \in \mathcal{T}.$$

Topology from an equivalence relation

Proposition

Let $(X, \mathcal{T})$ *be a topological space and* $f\colon X \to Y$ *be a surjective map. The system*

$$\mathcal{T}' = \{V \subseteq Y; f^{-1}[V] \in \mathcal{T}\}$$

is a topology on the set Y.

Definition of a quotient map

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and $f\colon (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a map. We say that f is a *quotient map* if f is surjective and

$$V \in \mathcal{T}_Y \qquad \Leftrightarrow \qquad f^{-1}[V] \in \mathcal{T}_X. \tag{1}$$

In this situation we also say that Y is a *quotient space* of the soace X.

Circle

$$S = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$$

- $f : I \to S,\ f(t) = (\cos 2\pi t, \sin 2\pi t)$
- $g : \mathbb{R} \to S,\ g(t) = (\cos 2\pi t, \sin 2\pi t)$

Cylinder

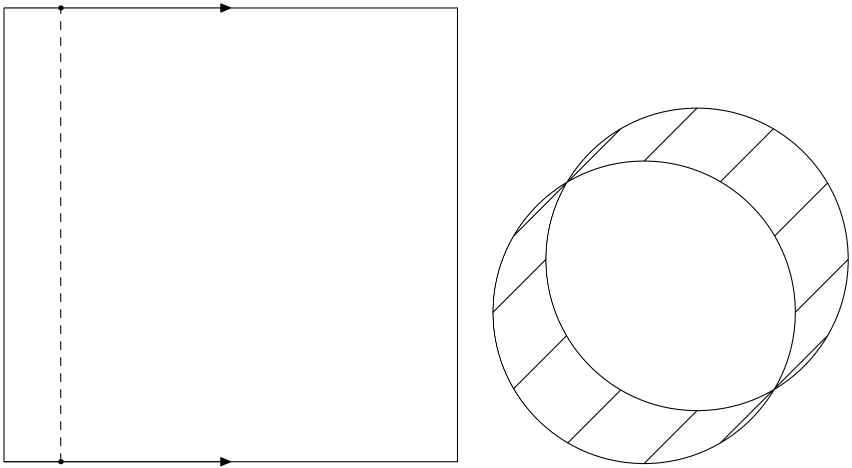
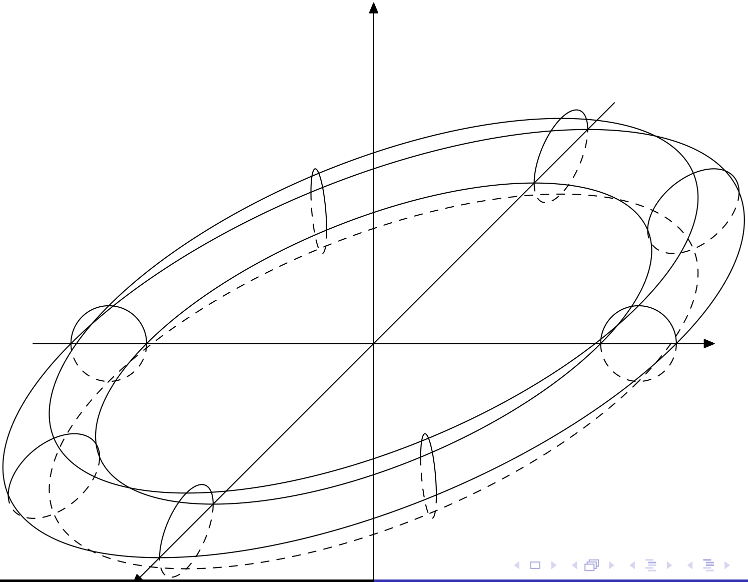
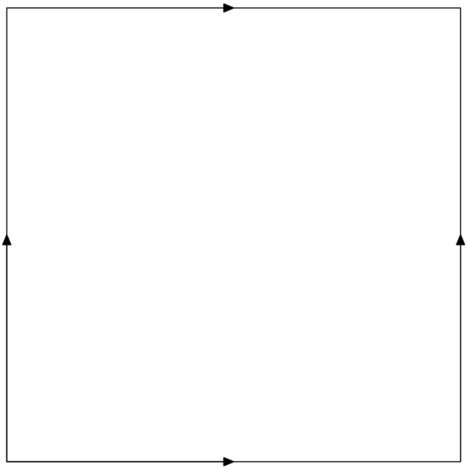


Figure: Cylinder as a quotient space of a square.

Torus



Torus



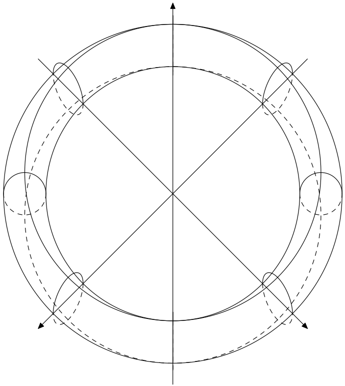


Figure: Torus as a quotient space of a square.

Möbius strip

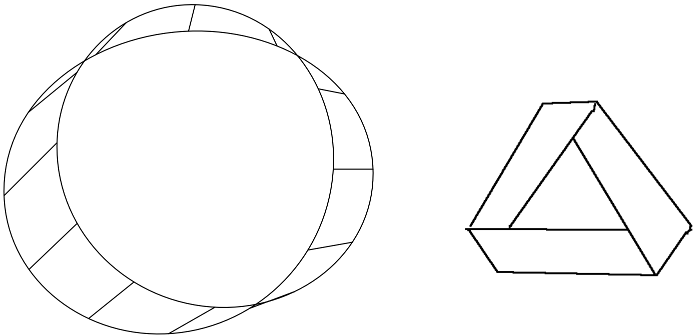


Figure: Möbius strip.

Möbius strip

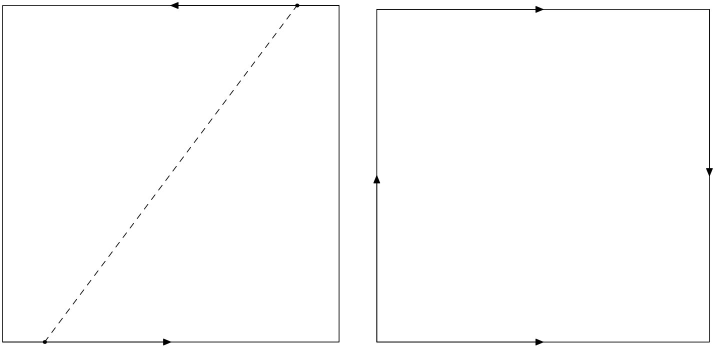
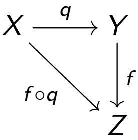


Figure: Möbius strip and Klein bottle can be obtained from a square

Quotient maps and continuity

Proposition

Let $q\colon X \to Y$ be a quotient map. Let Z be a topological space and $f\colon Y \to Z$. Then f is continuous iff the composition $f \circ q$ is continuous.



$$f \circ q^{-1}[V] = q^{-1}[f^{-1}[V]]$$

Preimages of closed sets

Proposition

Let $f\colon X \to Y$ be a surjective map between topological spaces. Then f is a quotient map iff for any $C \subseteq Y$ we have: The set $f^{-1}[C]$ is closed in X if and only if the set C is closed in Y.

$$f^{-1}[Y \setminus A] = X \setminus f^{-1}[A]$$

Composition of quotient maps

Proposition

If $f: X \to Y$ and $g: Y \to Z$ are quotient maps then also the composition $g \circ f: X \to Z$ is a quotient map.

$$V \in \mathcal{T}_Z \Leftrightarrow g^{-1}[V] \in \mathcal{T}_Y \Leftrightarrow (g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]] \in \mathcal{T}_X$$

Open and closed surjections

Proposition

If $f : X \to Y$ is surjective, continuous and open map then f is a quotient map.
If $f : X \to Y$ is surjective, continuous and closed map then f is a quotient map.

$$f[f^{-1}[A]] = A$$