Topological sum

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Topological sum

Four basic constructions

- subspace (embedding)
- quotient space (quotient map)
- **topological sum**
- product space

Topological sum

Definition

Let $(X_i, \mathcal{T}_i)$ be a topological space for each $i \in I$. We assume additionaly that the sets X_i are pairwise *disjoint*. Then we define a topology $\mathcal{T}$ on the set $X = \bigcup\limits_{i \in I} X_i$ as

$$\mathcal{T} = \{U \subseteq \bigcup_{i \in I} X_i; (\forall i \in I) U \cap X_i \in \mathcal{T}_i\},$$

i.e., the open sets are precisely the sets such that the intersection with X_i is open in X_i (for each $i \in I$).
The space $(X, \mathcal{T})$ is called the *topological sum* of the spaces $(X_i, \mathcal{T}_i)$ and denoted $\coprod\limits_{i \in I} X_i$.

Topological sum

Proposition

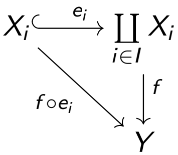
Let $\{X_i; i \in I\}$ *be a system of disjoint topological spaces and let* $X = \coprod_{i \in I} X_i$ *be their topological sum.*
Then every X_i *is a clopen subspace of* X_i.

Topological sum

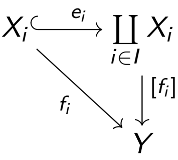
Proposition

Let $X = \coprod_{i \in I} X_i$ *and let* $e_i \colon X_i \hookrightarrow X$ *denotes the embedding of* X_i *into the topological sum. Let* $f \colon X \to Y$ *be a map into a topological space* Y*.*

The map f *is continuous if and only if* $f|_{X_i} = f \circ e_i$ *is continuous for every* $i \in I$*.*



Topological sum



Proposition

For every $i \in I$, let $f_i \colon X_i \to Y$ be a continuous map between topological spaces. Then also the map $[f_i] \colon \coprod_{i\in I} X_i \to Y$ is continuous.

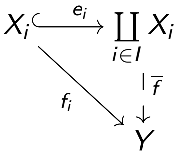
Topological sum

Proposition

Let $X = \coprod_{i \in I} X_i$ and let $e_i \colon X_i \hookrightarrow X$ be the embedding of X_i into the topological product. Let Y be a topological space and let $f_i \colon X_i \to Y$ be a continuous map for every $i \in I$.
Then there exists a unique continuous map $\overline{f} \colon \coprod_{i \in I} X_i \to Y$ such that

$$\overline{f} \circ e_i = f_i$$

holds for every $i \in I$.



Topological sum

$$h\colon \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$$

$$h(x) = f_i(x) \qquad \text{ak } x \in X_i$$

Topological sum

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ {\scriptstyle e_i}\downarrow & & \downarrow{\scriptstyle e_i'} \\ \coprod X_i & \xrightarrow[\coprod f_i]{} & \coprod Y_i \end{array}$$

Proposition

Let $f_i \colon X_i \to Y_i$ be a continuous map between topological spaces for every $i \in I$. Then also the map $\coprod_{i \in I} f_i \colon \coprod_{i \in I} X_i \to \coprod_{i \in I} Y_i$ is continuous.