Product space

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Product space

Four basic constructions

- subspace (embedding)
- quotient space (quotient map)
- topological sum
- **product space**

Product of two sets

$$X_1 \times X_2 = \{(x_1, x_2); x_1 \in X_1, x_2 \in X_2\}$$

$p_1 \colon X_1 \times X_2 \to X_1$ a $p_2 \colon X_1 \times X_2 \to X_2$

$$p_1(x_1, x_2) = x_1$$
$$p_2(x_1, x_2) = x_2$$

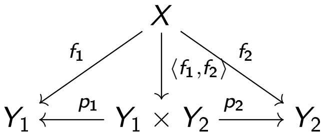
An alternative notation: $p_A \colon A \times B \to A$ where $p_A(a, b) = a$

Product of two sets

From $f_1 \colon X \to Y_1$ and $f_2 \colon X \to Y_2$ we get $g \colon X \to Y_1 \times Y_2$:

$$g(x) = (f_1(x), f_2(x)).$$

Notation: $\langle f_1, f_2 \rangle$.



Product of two sets

If we have $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$:

$$f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$$
$$(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$$

$$\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & Y_1 \times Y_2 \\ \downarrow p_i & & \downarrow p_i' \\ X_i & \xrightarrow[f_i]{} & Y_i \end{array}$$

If f_1, f_2 are injective (surjective, bijective) then $f_1 \times f_2$ injective (surjective, bijective), too.

Product of an arbitrary system

Definition

Suppose that for any $i \in I$ we are given a set X_i. The *Cartesian product* is defined as the system of all functions from I into $\bigcup_{i \in I} X_i$ such that $f(i) \in X_i$ holds for each $i \in I$.

$$\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i ; (\forall i \in I) f(i) \in X_i\}$$

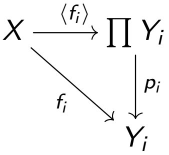
projections $p_i \colon \prod_{i \in I} X_i \to X_i$:

$$p_i(f) = f(i)$$

Product of an arbitrary system

If we have a map $f_i\colon X \to Y_i$ for each $i \in I$ then we get

$$\langle f_i \rangle \colon X \to \prod Y_i$$



Product of an arbitrary system

If we have a map $f_i\colon X_i \to Y_i$ for each $i \in I$ then we get

$$h = \prod_{i\in I} f_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$$

$$\begin{array}{ccc}
\prod X_i & \xrightarrow{\prod f_i} & \prod Y_i \\
{\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p_i'} \\
X_i & \xrightarrow[f_i]{} & Y_i
\end{array}$$

If all f_i's are injective (surjective, bijective) then $\prod\limits_{i\in I} f_i$ is injective (surjective, bijective) as well.

Definition of the product space

Definition

Let $(X_1, \mathcal{T}_1)$ a $(X_2, \mathcal{T}_2)$ be topological spaces. Let us define

$$\mathcal{B} = \{U \times V; U \in \mathcal{T}_1, V \in \mathcal{T}_2\}.$$

Then $\mathcal{B}$ is a base of a topology $\mathcal{T}$ on the Cartesian product $X_1 \times X_2$. this topology is called the *product topology* and the space $(X_1 \times X_2, \mathcal{T})$ is called the *product* of the spaces $(X_1, \mathcal{T}_1)$ a $(X_2, \mathcal{T}_2)$. Sometimes we will denote this topology as $\mathcal{T}_1 \times \mathcal{T}_2$.

$$\mathcal{B}' = \{U \times V; U \in \mathcal{B}_1, V \in \mathcal{B}_2\}$$

Definition of the product space

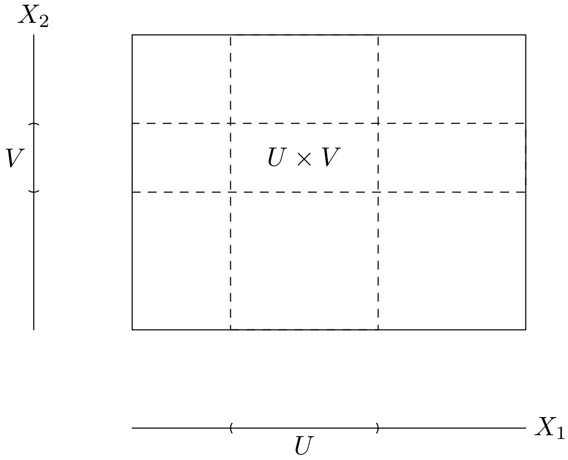


Figure: Basic sets in the product topology have the form $U \times V$

Examples

- Product of two discrete spaces is discrete.
- Product of two indiscrete spaces is indiscrete.
- Product of two metrizable spaces is metrizable.

$$d(x, y) = \max\{d(x_1, y_1), d(x_2, y_2)\}$$

Projekcie sú spojité a otvorené

Proposition

Let $X_1 \times X_2$ be the product of spaces X_1 and X_2. Let $p_i\colon X_1 \times X_2 \to X_i$ be the corresponding projections. The maps p_1 and p_2 are continuous and open.

A projection is not necessarily closed

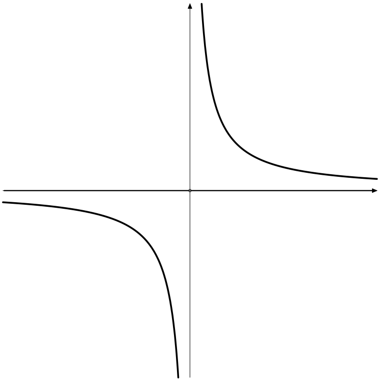


Figure: A projection is not necessarily closed

Product space and continuity

Proposition

Let Y, X_1, X_2 be topological spaces. A map $f\colon Y \to X_1 \times X_2$ is continuous iff $p_1 \circ f$ and $p_2 \circ f$ are continuous.

- $f_1 : X \to Y_1$, $f_2 : X \to Y_2$ continuous $\Rightarrow \langle f_1, f_2 \rangle$ continuous
- $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ continuous $\Rightarrow f_1 \times f_2$ continuous

Torus

Example

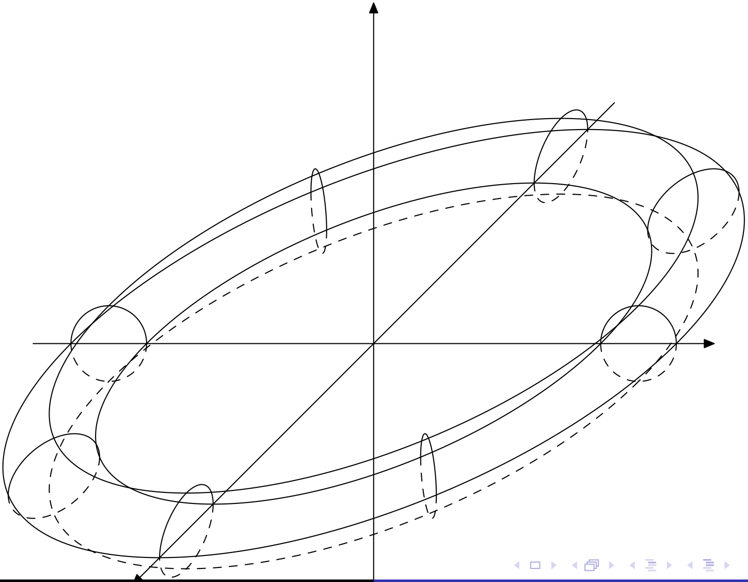
$$T = S \times S,$$

where S denotes a circle.
Quotient map $q\colon I \to S$, $q(t) = e^{i2\pi t}$ yields

$$q \times q\colon I \times I \to S \times S.$$

$(x, y) \sim (x', y') \Leftrightarrow \exp(i2\pi x) = \exp(i2\pi x')$ and $\exp(i2\pi y) = \exp(i2\pi y')$.

Torus



Definition of the product space

Definition

Let $(X_i, \mathcal{T}_i)$ be a topological space for each $i \in I$. Then

$$\mathcal{S} = \{p_i^{-1}[U]; i \in I, U \in \mathcal{T}_i\}$$

determines a subbase for a topology on $X = \prod_{i \in I} X_i$. Let us denote this topology as $\mathcal{T}$.

The space $(X, \mathcal{T})$ is called the *product* of the spaces $(X_i, \mathcal{T}_i)$ and denoted as $\prod_{i \in I} X_i$.

If $X_i = X$ for each $i \in I$, it is called the *power* of the space X and denoted X^I.

Definition of the product space

$$\mathcal{S} = \{p_i^{-1}[U]; i \in I, U \in \mathcal{T}_i\}$$

$$\mathcal{B} = \{\bigcap_{i \in F} p_i^{-1}[U_i]; U_i \in \mathcal{T}_i, F \text{ je konečná podmnožina množiny } I\}$$

$$= \{\bigcap_{i_1}^{i_k} p_{i_1}^{-1}[U_{i_1}] \cap \cdots \cap p_{i_k}^{-1}[U_{i_k}]; i_1, \ldots, i_k \in I, U_{i_j} \in \mathcal{T}_{i_j}\}$$

Definition of the product space

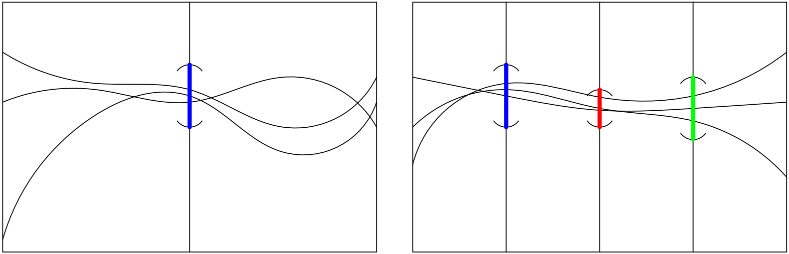


Figure: Illustration of a typic set from the subbase (or base) together with some functions from this set

Box topology

We do not want $\bigcap_{i\in I} p_i^{-1}[U_i] = \prod_{i\in I} U_i$.

- product of compact spaces
- universal property, categorial limit
- initial topology w.r.t. the projections
- characterization of continuity
- convergence = pointwise convergence

Continuity

Proposition

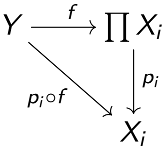
Let X_i be a topological space for each $i \in I$ and $\prod_{i \in I} X_i$ be the product space. Then the projection $p_i \colon \prod_{i \in I} X_i \to X_i$ is a continuous and open map for every $i \in I$.

Continuity

Proposition

Let Y be a topological space. Let X_i be a topological space for each $i \in I$. Let $f\colon Y \to \prod_{i \in I} X_i$..

The map f is continuous iff the composition $p_i \circ f$ is continuous for every $i \in I$.



Continuity

Corollary

Let $f_i\colon Y \to X_i$ be a continuous map between topological spaces for every $i \in I$. Then also $\langle f_i \rangle\colon Y \to \prod_{i\in I} X_i$ is continuous.

Corollary

Let $f_i\colon X_i \to Y_i$ be a continuous map between topological spaces for every $i \in I$. Then also $\prod_{i\in I} f_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ is continuous.

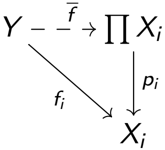
Universal property

Proposition

Let X_i be a topological space for every $i \in I$. Let us denote by $\prod_{i \in I} X_i$ the product space and by $p_i \colon X \to X_i$ the projections. Let Y be a topological space and let $f_i \colon Y \to X_i$ be a continuous map for every $i \in I$. Then there exists exactly one map $\overline{f} \colon Y \to \prod_{i \in I} X_i$ fulfilling

$$p_i \circ \overline{f} = f_i$$

for every $i \in I$. Moreover, $\overline{f}$ is continuous.



Closure in the product space

Proposition

Let X_i be a topological space and $A_i \subseteq X_i$ for every $i \in I$. Then

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

Corollary

Let C_i be a closed subset of X_i for every $i \in I$. Then also the product $\prod_{i \in I} C_i$ is closed in $\prod_{i \in I} X_i$

Countable base

- Product of countably many second countable spaces is second countable.
- Product of countably many first countable spaces is first countable. (I.e., there is a countable base for the product topology.)

Separable space

Theorem

Suppose the X_i is a separable space for each $i \in I$ and $|I| \leq \mathfrak{c}$. Then the product $\prod_{i \in I} X_i$ is a separable space.

Lemma

Let D be the discrete space with the cardinality $\aleph_0$ and let $|I| = \mathfrak{c}$. Then D^I is a separable space.