Initial and final topology

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Initial topology

Definition

For each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $f_i \colon X \to X_i$ be a function. Then the *initial topology* $\mathcal{T}$ (on the set X) *with respect to the system of functions* $\{f_i; i \in I\}$ is the coarsest topology such that every $f_i \colon (X, \mathcal{T}) \to (X_i, \mathcal{T}_i)$ is continuous.

Initial topology

Proposition

Let $\mathcal{T}$ be the initial topology on X w.r.t. the system of mappings $\{f_i \colon X \to X_i, i \in I\}$. Then

$$\mathcal{S} = \{f_i^{-1}[U]; i \in I, U \in \mathcal{T}_i\}$$

is a subbase for the initial topology.
Given any subbases $\mathcal{S}_i$ of the topologies $\mathcal{T}_i$, the system

$$\mathcal{S}' = \{f_i^{-1}[U]; i \in I, U \in \mathcal{S}_i\}$$

is also a subbase for the initial topology.

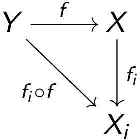
Examples of initial topology

- embedding
- product space
- subspace of a product
- weak and weak*-topology

Continuity

Theorem

Let X be a set let $(X_i, \mathcal{T}_i)$ be a topological space for each $i \in I$. Let $\mathcal{T}$ be the initial topology w.r.t. the system of mappings $\{f_i \colon X \to X_i, i \in I\}$.
For any topological space $(Y, \mathcal{T}')$ and any mapping $f \colon Y \to X$ we have: The mapping f is continuous if and only if the composition $f_i \circ f \colon Y \to X_i$ is continuous for every $i \in I$.



Initial topology and product

TODO

Final topology

Definition

Suppose that for each $i \in I$ we are given a topological space $(X_i, \mathcal{T}_i)$ and a function $f_i \colon X_i \to X$. Then the *final topology* $\mathcal{T}$ (on the set X) *with respect to the system of functions* $\{f_i; i \in I\}$ is the finest topology such that all functions $f_i \colon (X, \mathcal{T}) \to (X, \mathcal{T}_i)$ is continuous.

Final topology

Proposition

Suppose we have a system of functions $\{f_i\colon X_i \to X; i \in I\}$, *where* $(X_i, \mathcal{T}_i)$ *is a topological space for every* $i \in I$. *Le us define*

$$\mathcal{T} = \{U \subseteq X; (\forall i \in I) f_i^{-1}[U] \in \mathcal{T}_i\}.$$

Then $\mathcal{T}$ *is a topology on* X. *This topology is the final topology w.r.t. this system of functions.*

Examples of final topologies

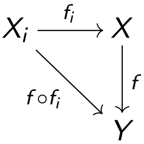
- faktorový priestor
- topologický súčet

$$X_i \overset{e_i}{\hookrightarrow} \coprod_{i\in I} X_i \xrightarrow{q} Y$$

Continuity

Theorem

For each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $\{f_i : X_i \to X; i \in I\}$ be a function. Let X as the final topology w.r.t. $\{f_i; i \in I\}$. Then a function $f : X \to Y$ is continuous if and only if $f \circ f_i$ is containuous for every $i \in I$.



Final topology

- The final topology w.r.t. the maps $f_i\colon X_i \to X$ is the same as the final topology w.r.t. $[f_i]\colon \coprod X_i \to X$.
- The function $[f_i]\colon \coprod X_i \to X$ is surjective iff $\bigcup_{i\in I} f[X_i] = X$.
- At the same time, the condition $\bigcup_{i\in I} f[X_i] = X$ describes when $[f_i]$ is a quotient map.