General topology – assignments version 1

For each of the problems you can get maximum of 20 points. It is good to start new solution on a separate page. If you want to write the solutions in LaTeX, it will appreciate if I get both the source code and the pdf.

Problem 1. Let $(X, \mathcal{T}_X)$, $(Y, \mathcal{T}_Y)$ be topological spaces and let $\mathcal{B}_X$ be a basis for the topology $\mathcal{T}_X$, $\mathcal{B}_Y$ be a basis for the topology $\mathcal{T}_Y$.

A function $f\colon X \to Y$ is continuous if and onvly if:
a) For each $V \in \mathcal{T}_Y$ we have $f^{-1}[V] \in \mathcal{T}_X$.
b) For every $V \in \mathcal{T}_Y$ and $x \in X$ such that $f(x) \in V$ there exists an $U \in \mathcal{T}_X$ such that $x \in U$ and $f[U] \subseteq V$.

Decide whether these equivalence remain valid if we replace each $\mathcal{T}$ with $\mathcal{B}$. (Include the reasoning for your claim.)

Problem 2. Let $(D, \leq_1)$ and $(E, \leq_2)$ be directed sets. On the set $D \times E$ we define the relation

$$(d_1, e_1) \leq (d_2, e_2) \Leftrightarrow (d_1 \leq_1 e_1) \wedge (d_2 \leq_2 e_2).$$

I.e. we compare any two pairs in such a way that the inequality holds for both coordinates.

Show that $(D \times E, \leq)$ is a directed set.

Problem 3. Let X be a topological space and let $\mathcal{F}$ be a filter on the set X. Prove that $\{(b, F); b \in F \in \mathcal{F}\}$ with

$$(b_1, F_1) \leq (b_2, F_2) \Leftrightarrow F_1 \supseteq F_2$$

is a directed set.

Let us define the net on this directed set by

$$x_{(b,F)} = b$$

Show that for any $a \in X$ we have:
a) This net converges to a if and only if $\mathcal{F} \to a$.
b) The point a is a cluster point of this net if and only if a is a cluster point of the filter $\mathcal{F}$.

Problem 4. Show that, if Urysohn's lemma holds for a topological space X, then X is normal.

I.e., we want to show that X is normal, if we are given that for any disjoint closed subsets A, B of this space there exists a continuous function $f\colon X \to \langle 0, 1 \rangle$ such that

$$A \subseteq f^{-1}[\{0\}] \qquad \text{a} \qquad B \subseteq f^{-1}[\{1\}].$$

Problem 5. Let $X = \{0, 1\}^\omega$ be the product of countably many copies of the discrete 2-point space and let $I = \langle 0, 1 \rangle$ be the closed unit interval with the usual topology. Let us define

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}.$$

a) Show that in this way we assigned to each $x \in X$ exactly one element $\varphi(x) \in I$. I.e., this is indeed $\varphi\colon X \to I$.
b) Show that the map φ is continuous.
c) Show that the map φ is surjective. Is it also injective?
d) Can you use these facts to prove compactness of $I = \langle 0, 1 \rangle$?