

# Binomické koeficienty

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# Definícia a základné vlastnosti

## Definícia

Pre ľubovoľné  $n \in \mathbb{N}_0$ ,  $k \in \mathbb{Z}$  definujeme *binomický koeficient*  $\binom{n}{k}$  ako počet všetkých  $k$ -prvkových podmnožín množiny  $n$ .

$$\begin{aligned}\binom{n}{k} &= \frac{n!}{(n-k)!k!} \\ \binom{n+1}{k} &= \binom{n}{k} + \binom{n}{k-1} \\ \binom{n}{k} &= \binom{n}{n-k}\end{aligned}$$

Pre reálne  $n$ 

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$
$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

## Binomická veta

$$(1 + t)^0 = 1$$

$$(1 + t)^1 = 1 + t$$

$$(1 + t)^2 = 1 + 2t + t^2$$

$$(1 + t)^3 = 1 + 3t + 3t^2 + t^3$$

$$(1 + t)^4 = 1 + 4t + 6t^2 + 4t^3 + t^4$$

$$(1 + t)^5 = 1 + 5t + 10t^2 + 10t^3 + 5t^4 + t^5$$

$$(1 + t)^6 = 1 + 6t + 15t^2 + 20t^3 + 15t^4 + 6t^5 + t^6$$

# Binomická veta

## Veta (Binomická veta)

Pre ľubovoľné  $t \in \mathbb{C}$  a  $n \in \mathbb{N}_0$  platí

$$(1+t)^n = \sum_{k=0}^n \binom{n}{k} t^k. \quad (1)$$

Pre ľubovoľné  $x, y \in \mathbb{C}$  a  $n \in \mathbb{N}_0$  platí

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}. \quad (2)$$

# Binomická veta

$$(1 + t)^n = \sum_{k=0}^n \binom{n}{k} t^k$$

- ▶ počet výberov pri roznásobovaní;
- ▶ dôkaz matematickou indukciou

## Binomická veta

$$\begin{aligned}(1+t)^{n+1} &= (1+t)(1+t)^n \\&= (1+t) \sum_{k=0}^n \binom{n}{k} t^k \\&= \sum_{k=0}^n \binom{n}{k} t^k + \sum_{k=1}^{n+1} \binom{n}{k-1} t^k \\&= \sum_{k=0}^{n+1} \left( \binom{n}{k} + \binom{n}{k-1} \right) t^k \\&= \sum_{k=0}^{n+1} \binom{n+1}{k} t^k\end{aligned}$$

## Binomická veta

$$\begin{aligned}
(1+t)^{n+1} &= (1+t)(1+t)^n \\
&= (1+t) \sum_{k=0}^n \binom{n}{k} t^k \\
&= \sum_{k=0}^n \binom{n}{k} t^k + t \sum_{k=0}^n \binom{n}{k} t^k \\
&= \sum_{k=0}^n \binom{n}{k} t^k + \sum_{k=0}^n \binom{n}{k} t^{k+1} \\
&= \sum_{k=0}^n \binom{n}{k} t^k + \sum_{k=1}^{n+1} \binom{n}{k-1} t^k \\
&\stackrel{(*)}{=} \sum_{k=0}^{n+1} \binom{n}{k} t^k + \sum_{k=0}^{n+1} \binom{n}{k-1} t^k \\
&= \sum_{k=0}^{n+1} \left( \binom{n}{k} + \binom{n}{k-1} \right) t^k \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} t^k
\end{aligned}$$

# Striedavé znamienka

## Dôsledok

Pre ľubovoľné  $n \in \mathbb{N}_0$  také, že  $n \geq 1$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

A teda platia aj rovnosti

$$\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2j+1} = 2^{n-1}.$$

## Párne a nepárne

$$\sum_{0 \leq 2j \leq n} \binom{n}{2j} = \sum_{0 \leq 2j+1 \leq n} \binom{n}{2j+1}$$
$$\sum_{j=0}^{\infty} \binom{n}{2j} = \sum_{j=0}^{\infty} \binom{n}{2j+1} = 2^{n-1}$$

Kombinatorický argument, že máme rovnako veľa párnych a nepárnych podmnožín?

# Párne a nepárne

$$\begin{array}{r}
 (1+1)^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{2k} + \binom{n}{2k+1} + \cdots \\
 (1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \cdots + \binom{n}{2k} - \binom{n}{2k+1} + \cdots \\
 \hline
 2^n = 2\binom{n}{0} + 2\binom{n}{2} + \cdots + 2\binom{n}{2k} + \cdots
 \end{array}$$

# Násobky trojky a štvorky

$$\sum_{k=0}^{\infty} \binom{n}{3k} = ?$$

$$\sum_{k=0}^{\infty} \binom{n}{4k} = ?$$

## Derivácia

$$f(t) = (1+t)^n = \sum_{k=0}^n \binom{n}{k} t^k$$

$$f'(t) = n(1+t)^{n-1}$$

$$f'(t) = \sum_{k=1}^n k \binom{n}{k} t^{k-1}$$

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

## Iné odvodenie

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

$$\sum_{k=0}^n k \binom{n}{k} = \sum_{k=0}^n (n-k) \binom{n}{n-k}$$

## Iné odvodenie

$$\sum_{k=0}^n k \binom{n}{k} = n2^{n-1}$$

$$S = \sum_{k=0}^n k \binom{n}{k}$$

$$S = \sum_{k=0}^n (n-k) \binom{n}{k}$$

$$2S = n \sum_{k=0}^n \binom{n}{k} = n2^n$$

## Integrovanie

$$f(t) = (1 + t)^n = \sum_{k=0}^n \binom{n}{k} t^k$$

$$\int_0^x f(t) \, dt = \frac{(1+x)^{n+1}}{n+1} = \sum_{k=0}^n \binom{n}{k} \frac{x^{k+1}}{k+1}$$

$$\sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}$$

## Vlastnosti binomických koeficientů

$$k \binom{n}{k} = n \binom{n-1}{k-1} \quad (3)$$

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1} \quad (4)$$

# Vlastnosti binomických koeficientov

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

- ▶ Výber  $k$ -člennej komisie a predsedu.
- ▶  $\{(x, A); x \in A, A \subseteq M, |A| = k\}$

# Vlastnosti binomických koeficientů

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$
$$\binom{n}{k} \binom{k}{l} = \binom{n}{l} \binom{n-l}{k-l}$$

## Vlastnosti binomických koeficientov

$$\binom{n}{k} < \binom{n}{k+1} \quad \text{ak } 2k+1 < n$$

$$\binom{n}{k} > \binom{n}{k+1} \quad \text{ak } 2k+1 > n$$

# Vandermondova identita

## Tvrdenie

Pre ľubovoľné  $m, n \in \mathbb{N}_0$  platí

$$\sum_{j=0}^s \binom{m}{j} \binom{n}{s-j} = \binom{m+n}{s}. \quad (5)$$

## Zovšeobecnenie binomického koeficientu

## Definícia

Pre  $t \in \mathbb{C}$  a  $k \in \mathbb{N}_0$  označme

$$\binom{t}{k} = \frac{t(t-1)\cdots(t-k+1)}{k!}$$

$$\binom{t}{0} = 1$$

# Zovšeobecnenie binomickej vety

## Veta

*Nech  $t \in \mathbb{R}$ . Nech  $x \in \mathbb{R}$ ,  $|x| < 1$ . Potom platí rovnosť*

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k. \quad (6)$$

*(T.j. rad na pravej strany hodnosti konverguje k hodnote uvedenej na ľavej strane.)*

## Zovšeobecnenie binomickej vety

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k$$

$$(1+x)^t = 1 + tx + \frac{t(t-1)}{2!}x^2 + \frac{t(t-1)(t-2)}{3!}x^3 + \dots +$$

$$+ \frac{t(t-1)(t-2)\dots(t-k+1)}{k!}x^k + \dots$$

Taylorov rad funkcie  $f(x) = (1+x)^t$  v nule.

Taylorov rad pre  $f(x) = (1+x)^t$ 

$$f(x) = (1+x)^t$$

$$f^{(k)}(x) = t(t-1)\cdots(t-k+1)x^{t-k}$$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(k)}(0)}{k!}x^k + \cdots$$

$$(1+x)^t = 1 + tx + \frac{t(t-1)}{2}x^2 + \cdots + \binom{t}{k}x^k + \cdots$$

$$(1+x)^t = \sum_{k=0}^{\infty} \binom{t}{k} x^k$$

$\binom{n}{k}$  pre záporné  $n$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}.$$

$$\begin{aligned} \binom{-n}{k} &= \frac{(-n)(-n-1)\cdots(-n-k+2)(-n-k+1)}{k!} \\ \binom{n+k-1}{k} &= \frac{(n+k-1)(n+k-2)\cdots(n+1)n}{k!} \end{aligned}$$

Prípad  $n = -1$ 

$$\binom{-1}{k} = (-1)^k$$

$$(1+x)^{-1} = \sum_{k=0}^{\infty} (-1)^k x^k$$

$$(1-x)^{-1} = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

## Záporný exponent

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k}$$

$$(1+x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} x^k = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

# Záporný exponent

$$\frac{1}{(1-x)^n} = (1+x+x^2+x^3+\dots)^n.$$

$$a_1 + a_2 + \dots + a_n = k.$$

$$\binom{n+k-1}{k}$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-x)^k = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k$$

## Multinomická veta

$$(x_1 + x_2 + \cdots + x_k)^n = ?$$

$$\underbrace{(x_1 + x_2 + \cdots + x_k)(x_1 + x_2 + \cdots + x_k) \cdots (x_1 + x_2 + \cdots + x_k)}_{n\text{-krát}}$$

Koľkými spôsobmi môžeme dostať  $x_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$ ?

$$\binom{n}{a_1, a_2, \dots, a_k} = \frac{n!}{a_1! a_2! \cdots a_k!}$$

# Multinomická veta

## Veta (Multinomická veta)

Pre ľubovoľné  $x_1, x_2, \dots, x_k \in \mathbb{R}$  platí rovnosť

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{\substack{a_1 + \dots + a_k = n \\ a_i \geq 0}} \binom{n}{a_1, a_2, \dots, a_k} x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}$$