Fibonacciho čísla

1. júla 2024

Fibonacciho čísla

Definícia

Pre $n \in \mathbb{N}_0$ definujeme *Fibonacciho číslo* F_n rekurentne, pomocou podmienok $F_0 = 0$, $F_1 = 1$ a

$$F_{n+2} = F_{n+1} + F_n. \tag{1}$$

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34

n	10	11	12	13	14	15	16	17	18	19
F_n	55	89	144	233	377	610	987	1597	2584	4181

Záporné indexy

$$F_{n+2} = F_{n+1} + F_n$$

Niekedy sa nám môžem hodiť pracovať s F_n aj pre záporné indexy n.

n	1	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
F_n	1	0	1	-1	2	-3	5	-8	13	-21	34

Záporné indexy

Niekedy sa nám môžem hodiť pracovať s F_n aj pre záporné indexy n.

n	0	-1	-2	-3	-4	-5	-6	-7	-8	-9
F_n	0	1	-1	2	-3	5	-8	13	-21	34

$$F_{-n} = (-1)^n F_n$$

Binetov vzorec

F_n vyjadríme pomocou

$$\varphi = \frac{1+\sqrt{5}}{2}$$
$$\psi = \frac{1-\sqrt{5}}{2}$$

t.j. pomocou koreňov rovnice $x^2 - x - 1 = 0$.

$$\varphi + \psi = 1$$
$$\varphi - \psi = \sqrt{5}$$
$$\varphi \cdot \psi = -1$$

Binetov vzorec

Tvrdenie

Pre ľubovoľné $n \in \mathbb{N}_0$ *platí:*

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}} \tag{2}$$

Binetov vzorec

$$
\begin{aligned}
F_{n+2} &= F_{n+1} + F_n \\
&= \frac{\varphi^{n+1} - \psi^{n+1}}{\sqrt{5}} + \frac{\varphi^n - \psi^n}{\sqrt{5}} \\
&= \frac{\varphi^{n+1} + \varphi^n}{\sqrt{5}} - \frac{\psi^{n+1} + \psi^n}{\sqrt{5}} \\
&= \frac{(\varphi + 1)\varphi^n}{\sqrt{5}} - \frac{(\psi + 1)\psi^n}{\sqrt{5}} \\
&= \frac{\varphi^2 \cdot \varphi^n}{\sqrt{5}} - \frac{\psi^2 \cdot \psi^n}{\sqrt{5}} \\
&= \frac{\varphi^{n+2} - \psi^{n+2}}{\sqrt{5}}
\end{aligned}
$$

Binetov vzorec

$$\varphi^{n+2} = \varphi^{n+1} + \varphi^{n}$$
$$\psi^{n+2} = \psi^{n+1} + \psi^{n}$$

Binetov vzorec

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi = \frac{1+\sqrt{5}}{2}$$

Fibonacciho čísla a matice

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} \tag{3}$$

$$\begin{pmatrix} A_{n+1} \\ A_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} A_1 \\ A_0 \end{pmatrix} \tag{4}$$

Fibonacciho čísla a matice

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = A + I$$

$$\chi_A(t) = t^2 - \mathrm{Tr}(A)t + \det(A) = t^2 - t - 1$$

Cassiniho identita

Tvrdenie (Cassiniho identita)

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n \tag{5}$$

$$\det\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

Cassiniho identita

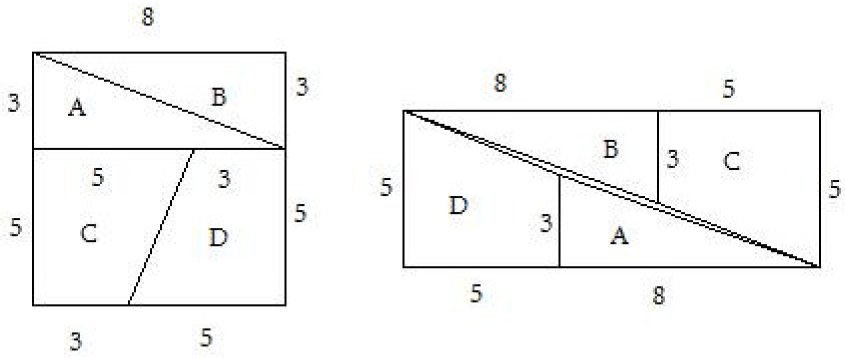

Figure: Cassiniho identita.

Cassiniho identita

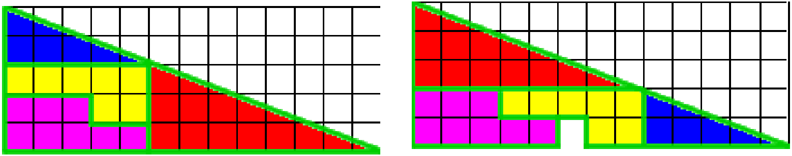

Figure: Missing square puzzle

Cassiniho identita

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

$$\frac{F_{n+1}}{F_n} - \frac{F_n}{F_{n-1}} = \frac{(-1)^n}{F_nF_{n-1}}$$

Diagonalizácia a Binetov vzorec

$$A = P \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} P^{-1}$$

$$A^n = P \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} P^{-1}$$

Diagonalizácia a Binetov vzorec

$$\begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = P \begin{pmatrix} \varphi^n & 0 \\ 0 & \psi^n \end{pmatrix} P^{-1}$$

$$F_n = c_1 \varphi^n + c_2 \psi^n$$

Diagonalizácia a Binetov vzorec

$$\begin{aligned} c_1\varphi^n + c_2\psi^n &= F_n \\ c_1 + c_2 &= 0 \\ c_1\varphi + c_2\psi &= 1 \end{aligned}$$

$$c_1 = \frac{\det\begin{pmatrix} 0 & 1 \\ 1 & \psi \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ \varphi & \psi \end{pmatrix}} = \frac{-1}{\psi - \varphi} = \frac{1}{\sqrt{5}}$$

$$c_2 = \frac{\det\begin{pmatrix} 1 & 0 \\ \varphi & 1 \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 \\ \varphi & \psi \end{pmatrix}} = \frac{1}{\psi - \varphi} = -\frac{1}{\sqrt{5}}$$

Diagonalizácia a Binetov vzorec

$$F_n = \frac{\varphi^n - \psi^n}{\varphi - \psi} = \frac{\varphi^n - \psi^n}{\sqrt{5}}$$

Kombinatorická interpretácia Fibonacciho čísel

- Počet dláždení mriežky rozmerov $1 \times n$ dlaždicami veľkostí 1×1 a 1×2.
- Počet vyjadrení čísla n ako súčtu jednotiek a dvojok.
- Chceme prejsť n schodov tak, že každým krokom stúpame o jeden alebo o dva schody vyššie. Koľko je možností, ako sa to dá urobiť?

Kombinatorická interpretácia Fibonacciho čísel

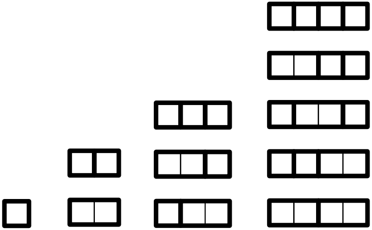

Figure: Dláždenia mriežky $1 \times n$ pre $n = 1, 2, 3, 4$

Kombinatorická interpretácia Fibonacciho čísel

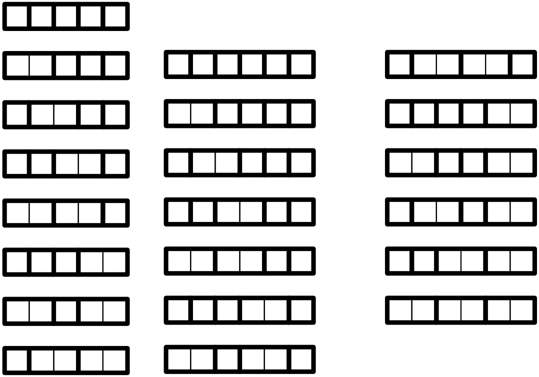

Figure: Dláždenia mriežky $1 \times n$ pre $n = 5, 6$

Kombinatorická interpretácia Fibonacciho čísel

Tvrdenie

Počet dláždení mriežky rozmerov $1 \times n$ *pomocou dlaždíc rozmerov* 1×1 *a* 1×2 *je rovný* F_{n+1}.

Konvolučná vlastnosť

Tvrdenie (Konvolučná vlastnosť)

Pre ľubovoľné $m, n \in \mathbb{N}_0$ platí

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n \tag{6}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^m \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{m+n}$$

$$\begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix}$$

Súčet prvých n Fibonacciho čísel

$$S_n = \sum_{k=0}^{n} F_n$$

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
S_n	0	1	2	4	7	12	20	33	54	88

Súčet prvých n Fibonacciho čísel

$$S_n = \sum_{k=0}^{n} F_n$$

n	0	1	2	3	4	5	6	7	8	9
F_n	0	1	1	2	3	5	8	13	21	34
S_n	0	1	2	4	7	12	20	33	54	88

$$\sum_{k=0}^{n} F_n = F_{n+2} - 1$$

Suma $F_{n+1} = \sum_{k=0}^{\infty} \binom{n-k}{k}$

$$F_{n+1} = \sum_{k=0}^{\infty} \binom{n-k}{k} \tag{7}$$

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \dots$$

Suma $F_{n+1} = \sum_{k=0}^{\infty} \binom{n-k}{k}$

$$\binom{0}{0} = 1 = F_1$$

$$\binom{1}{0} = 1 = F_2$$

$$\binom{2}{0} + \binom{1}{1} = 1 + 1 = 2 = F_3$$

$$\binom{3}{0} + \binom{2}{1} = 1 + 2 = 3 = F_4$$

$$\binom{4}{0} + \binom{3}{1} + \binom{2}{2} = 1 + 3 + 1 = 5 = F_5$$

$$\binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8 = F_6$$

$$\binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13 = F_7$$

$$\binom{7}{0} + \binom{6}{1} + \binom{5}{2} + \binom{4}{3} = 1 + 6 + 10 + 4 = 21 = F_8$$

Suma $F_{n+1} = \sum_{k=0}^{\infty} \binom{n-k}{k}$

$$
\begin{array}{cccccccc}
\binom{0}{0} & & & & & & & \\
\binom{1}{0} & \binom{1}{1} & & & & & & \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} & & & & & \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} & & & & \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & & & \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5} & & \\
\binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \binom{6}{3} & \binom{6}{4} & \binom{6}{5} & \binom{6}{6} & \\
\binom{7}{0} & \binom{7}{1} & \binom{7}{2} & \binom{7}{3} & \binom{7}{4} & \binom{7}{5} & \binom{7}{6} & \binom{7}{7}
\end{array}
$$

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	