Topological spaces

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Metric spaces

Definition

Let X be a set. A function $d\colon X \times X \to \mathbb{R}$ is a *metric* if, for any $x, y \in X$, we have

(D1). $d(x, y) \geq 0$;

(D2). $d(x, y) = 0$ práve vtedy, keď $x = y$;

(D3). $d(x, y) = d(y, x)$

(D4). $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a metric on X, the pair (X, d) is called a *metric space*.

Metric spaces

- closure = limits of sequences
- continuity = sequential continuity
- compactness via existence of convergent subsequences
- Every continuous function has a minimum and a maximum.

Topological spaces

Relevant notions this semester:

- continuity
- convergence
- compactness

sequences $\longrightarrow$ nets, filters

An application – compactness and approximations

$$\ell_1 \subsetneq \ell_\infty^*$$

There exists a functional $\varphi \in \ell_\infty^*$ extending the usual limit.

- $\varphi \in \ell_\infty^* \setminus \ell_1$.
- This can be shown using Hahn-Banach Theorem.
- Here: a proof using compactness and convergence.

An application – compactness and approximations

$$\varphi_i \colon \ell_\infty \to \mathbb{R}$$
$$\varphi_i \colon x = (x_n)_{n=0}^{\infty} \mapsto x_i$$

- They belong to the unit ball of ℓ_∞^*, considered with weak* topology this is a compact space.
- Limit of a convergent subnet: $\varphi = \lim_{d \in D} \varphi_{n_d}$

An application – compactness and approximations

$$\varphi = \lim_{d \in D} \varphi_{n_d}$$

- φ is linear
- $\|\varphi_n\| = 1 \Rightarrow \|\varphi\| = 1$
- $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$
- $n_d \xrightarrow[d \in D]{} \infty$ implies $\varphi(e^{(i)}) = 0$

$$\varphi \in \ell_\infty^* \setminus \ell_1$$