

Quotient maps

October 21, 2024

Equivalence relations and surjections

- ▶ A surjection $f: X \rightarrow Y$ induces the equivalence relation

$$x_1 \sim x_2 \Leftrightarrow f(x_1) = f(x_2),$$

the equivalence classes are $f^{-1}(y)$.

- ▶ If \sim is an equivalence relation then for the quotient set $X/\sim = \{[x]; x \in X\}$ we have a surjection

$$\begin{aligned} p: X &\rightarrow X/\sim \\ p: x &\mapsto [x] \end{aligned}$$

- ▶ Hence any equivalence relation can be determined by a surjective function and, conversely, every equivalence relation gives a canonical surjective function.

Topology from an equivalence relation

Proposition

Let (X, \mathcal{T}) be a topological space and \sim be an equivalence relation on the set X . Let us denote $p: X \rightarrow X/\sim$ the map given by $p(x) = [x]$. Let us define

$$\mathcal{T}_{\sim} = \{V \subseteq X/\sim; p^{-1}[V] \in \mathcal{T}\}.$$

Then \mathcal{T}_{\sim} is a topology on the set X/\sim .

Topologický priestor $(X/\sim, \mathcal{T}_{\sim})$ budeme nazývať faktorový priestor priestoru X podľa relácie \sim .

$p: X \rightarrow X/\sim$ je surjekcia a platí:

$$V \in \mathcal{T}_{\sim} \quad \Leftrightarrow \quad p^{-1}[V] \in \mathcal{T}.$$

Topology from an equivalence relation

Proposition

Let (X, \mathcal{T}) be a topological space and $f: X \rightarrow Y$ be a surjective map. The system

$$\mathcal{T}' = \{V \subseteq Y; f^{-1}[V] \in \mathcal{T}\}$$

is a topology on the set Y .

Definition of a quotient map

Definition

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces and $f: (X, \mathcal{T}_X) \rightarrow (Y, \mathcal{T}_Y)$ be a map. We say that f is a *quotient map* if f is surjective and

$$V \in \mathcal{T}_Y \quad \Leftrightarrow \quad f^{-1}[V] \in \mathcal{T}_X. \quad (1)$$

In this situation we also say that Y is a *quotient space* of the space X .

Circle

$$S = \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$$

- ▶ $f: I \rightarrow S, f(t) = (\cos 2\pi t, \sin 2\pi t)$
- ▶ $g: \mathbb{R} \rightarrow S, g(t) = (\cos 2\pi t, \sin 2\pi t)$

Cylinder

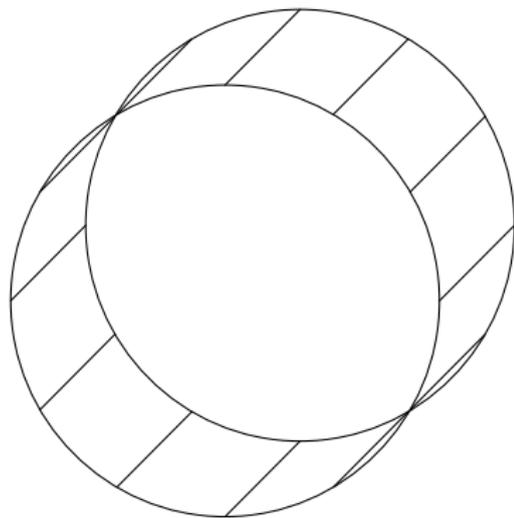
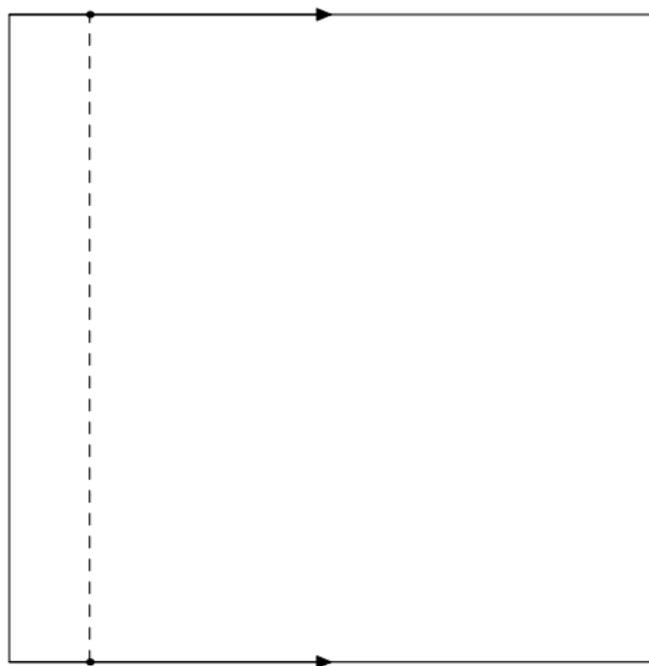
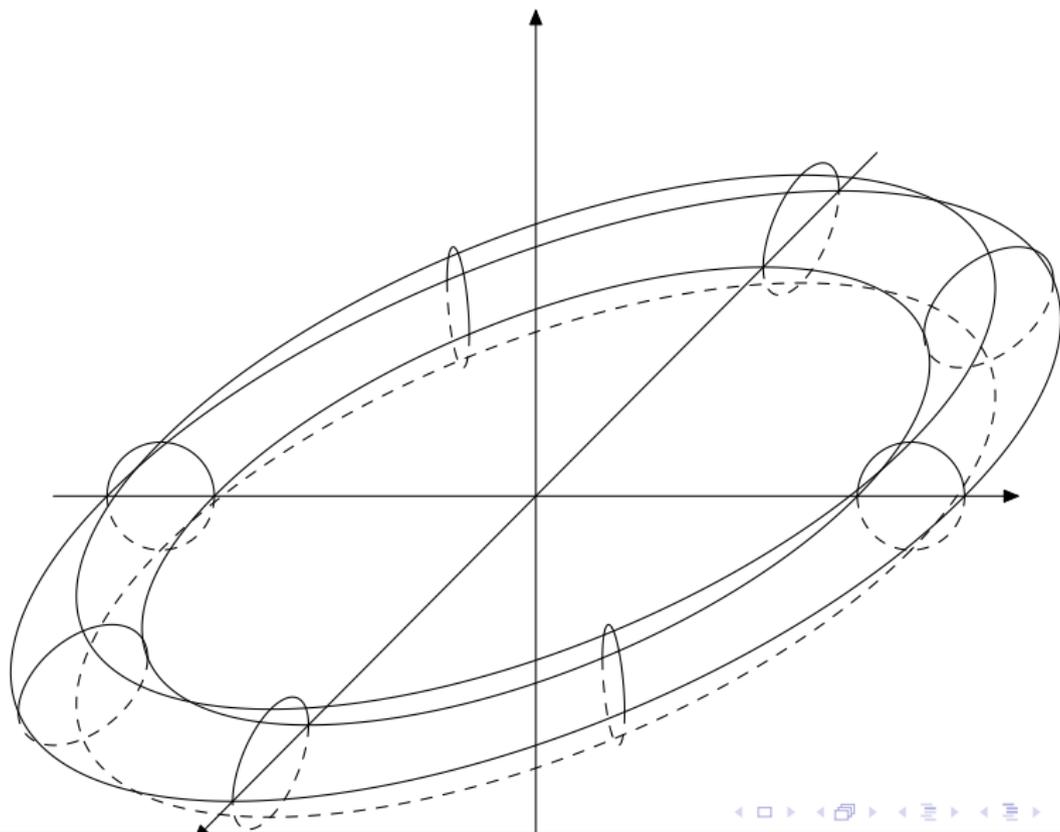


Figure: Cylinder as a quotient space of a square.

Torus



Torus

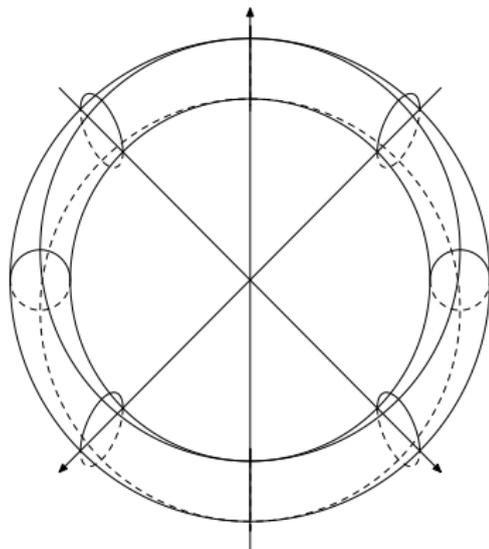
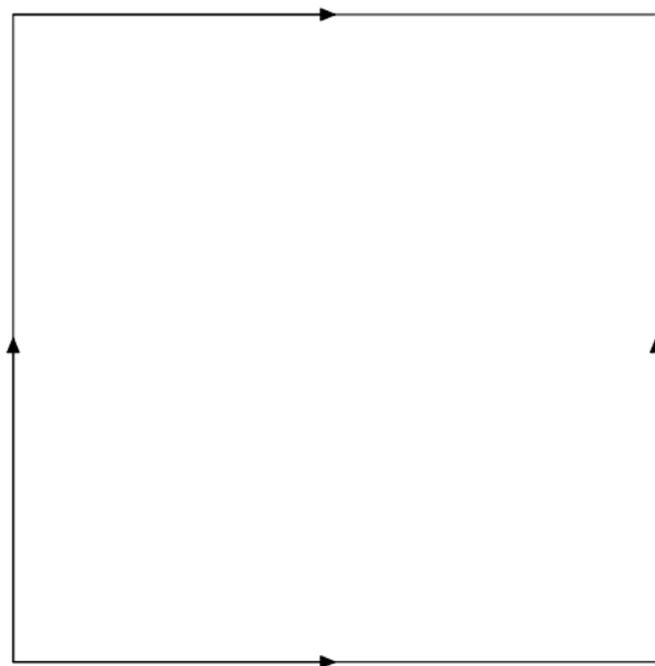


Figure: Torus as a quotient space of a square.

Möbius strip

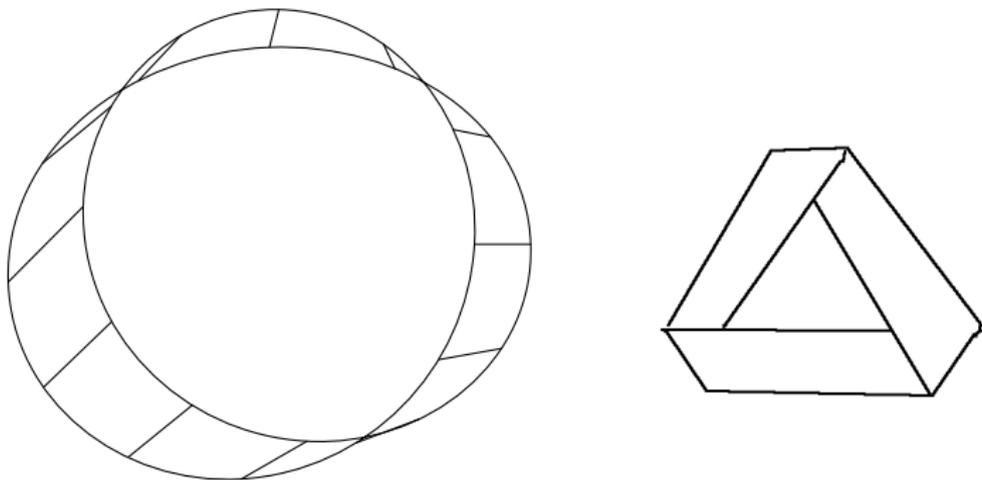


Figure: Möbius strip.

Möbius strip

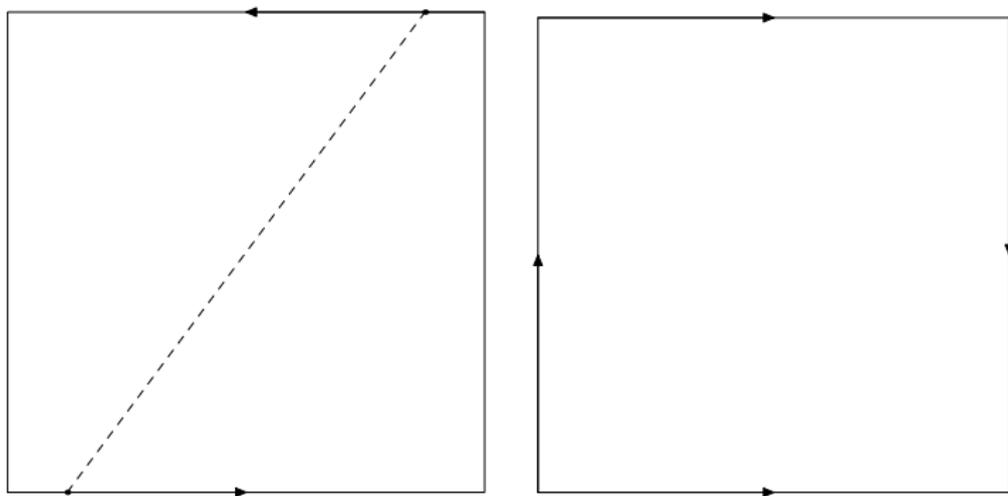


Figure: Möbius strip and Klein bottle can be obtained from a square

Quotient maps and continuity

Proposition

Let $q: X \rightarrow Y$ be a quotient map. Let Z be a topological space and $f: Y \rightarrow Z$. Then f is continuous iff the composition $f \circ q$ is continuous.

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ & \searrow f \circ q & \downarrow f \\ & & Z \end{array}$$

$$f \circ q^{-1}[V] = q^{-1}[f^{-1}[V]]$$

Preimages of closed sets

Proposition

Let $f: X \rightarrow Y$ be a surjective map between topological spaces. Then f is a quotient map iff for any $C \subseteq Y$ we have: The set $f^{-1}[C]$ is closed in X if and only if the set C is closed in Y .

$$f^{-1}[Y \setminus A] = X \setminus f^{-1}[A]$$

Composition of quotient maps

Proposition

If $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ are quotient maps then also the composition $g \circ f: X \rightarrow Z$ is a quotient map.

$$V \in \mathcal{T}_Z \Leftrightarrow g^{-1}[V] \in \mathcal{T}_Y \Leftrightarrow (g \circ f)^{-1}[V] = f^{-1}[g^{-1}[V]] \in \mathcal{T}_X$$

Open and closed surjections

Proposition

If $f: X \rightarrow Y$ is surjective, continuous and open map then f is a quotient map.

If $f: X \rightarrow Y$ is surjective, continuous and closed map then f is a quotient map.

$$f[f^{-1}[A]] = A$$

Compactness and closed maps

Neskôr budeme mať:

- ▶ Continuous image of a compact space is compact.
- ▶ If X is a compact space, Y is a Hausdorff space and $f: X \rightarrow Y$ is continuous, then f is a closed map.
- ▶ For example, $I = \langle 0, 1 \rangle$ is compact.