General topology – assignments version 2

For each of the problems you can get maximum of 20 points. It is good to start new solution on a separate page. If you want to write the solutions in LaTeX, it will appreciate if I get both the source code and the pdf.

Úloha 1. Let $q\colon X \to Y$ be a continuous surjective function. Show that if, for every topological space Z and every function $f\colon Y \to Z$, continuity of $f \circ q$ implies continuity f then q is a quotient map.

Úloha 2. Let $(D, \leq)$ be a directed set and $x\colon D \to \mathbb{R}$ satisfies

$$(\forall d, d' \in D)(d \leq d' \Rightarrow x_d \leq x_{d'}).$$

(I.e., it is a non-decreasing net consisting of real numbers.)
    Show that if this net is bounded from above then there exists a limit $L = \lim\limits_{d \in D} x_d$ and the equality

$$L = \sup_{d \in D} x_d$$

holds.

Úloha 3. Show that a topological space $(X, \mathcal{T})$ is normal if and only if the following condition holds: For any open sets $U, V \subseteq X$ such that $U \cup V = X$, there exist closed sets $C, D \subseteq X$ fulfilling $C \subseteq U$, $D \subseteq V$ and $C \cup D = X$.

Úloha 4. Show that if Tietze theorem holds for a topological space X then X is normal.
    I.e., we want to show, that X is normal if, for any closed subset $A \subseteq X$ and any continuous map $f\colon A \to \mathbb{R}$, there exists a continuous map $F\colon X \to \mathbb{R}$ such that

$$F|_A = f.$$

Úloha 5. Let $X = \{0,1\}^\omega$ be the product of countably many copies of the discrete 2-point space and let $I = \langle 0, 1 \rangle$ be the closed unit interval with the usual topology. Let us define

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}.$$

a) Show that in this way we assigned to each $x \in X$ exactly one element $\varphi(x) \in I$. I.e., this is indeed $\varphi\colon X \to I$.
b) Show that the map φ is continuous.
c) Show that the map φ is surjective. Is it also injective?
d) Can you use these facts to prove compactness of $I = \langle 0, 1 \rangle$?