

Topological spaces

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Definition of topology

Definition

Let X be a set. A system $\mathcal{T} \subseteq \mathcal{P}(X)$ is called a *topology* on the set X if:

(O1). $\emptyset, X \in \mathcal{T}$.

(O2). If $A, B \in \mathcal{T}$, then also $A \cap B \in \mathcal{T}$.

(O3). if $A_i \in \mathcal{T}$ for each $i \in I$, then also $\bigcup_{i \in I} A_i \in \mathcal{T}$.

If $\mathcal{T}$ is a topology on X , then the pair $(X, \mathcal{T})$ is called a *topological space*. The sets belonging to $\mathcal{T}$ are called *open sets* in the space $(X, \mathcal{T})$.

Definition of topology

- ▶ There are various ways to describe a topology – this axiomatization is relatively simple.
- ▶ One possible viewpoint is that we are “measuring” distance – but using bigger and smaller neighborhood rather than using numbers.

Examples of topologies

- ▶ $\mathcal{T}_{ind} = \{\emptyset, X\} = \text{indiscrete topology}$
- ▶ $\mathcal{T}_{disc} = \mathcal{P}(X) = \text{discrete topology}$
- ▶ Sierpiński space

Metric spaces

Definition

Let X be a set. A function $d: X \times X \rightarrow \mathbb{R}$ is a *metric* if, for any $x, y \in X$, we have

(D1). $d(x, y) \geq 0$;

(D2). $d(x, y) = 0$ práve vtedy, keď $x = y$;

(D3). $d(x, y) = d(y, x)$

(D4). $d(x, z) \leq d(x, y) + d(y, z)$.

If d is a metric on X , the pair (X, d) is called a *metric space*.

Topology given by a metric

$$B(a, r) = \{x \in X; d(x, a) < r\}$$

- ▶ A point $a \in U$ is called an *interior point* of a set U if there is a real number $r > 0$ such that $B(a, r) \subseteq U$.
- ▶ If each point of a set U is its interior point we say that U is *open in the metric space* (X, d) .

$$(X, \mathcal{T}_d)$$

Subspace

Definition

Let $(X, \mathcal{T})$ be a topological space and $S \subseteq X$. If we define

$$\mathcal{T}_S = \{U \cap S; U \in \mathcal{T}\},$$

then $\mathcal{T}_S$ is a topology on S .

The pair $(S, \mathcal{T}_S)$ is called a *subspace* of the topological space X .

The topology $\mathcal{T}_S$ is called the *relative topology*

If S is an open subset of X , we say that it is an *open subspace*.

Similarly, S is called a *closed subspace* if S is a closed subset.

Closed sets

Definition

Let X be a topological space and $C \subseteq X$.

A subset C is called a *closed set*, if the complement $X \setminus C$ is an open set.

If C is both closed and open in X , we say that it is a *clopen set*.

Sets are not doors. They can be open, closed, both, or neither.

Neznámy autor

Closed sets

Proposition

Let $(X, \mathcal{T})$ be a topological space. Let $\mathcal{C}$ be the system of all closed sets in X . Then the following holds:

(C1). $\emptyset, X \in \mathcal{C}$.

(C2). Ak $A, B \in \mathcal{C}$ tak, $A \cup B \in \mathcal{C}$.

(C3). Ak $A_i \in \mathcal{C}$ pre všetky $i \in I$ (pričom $I \neq \emptyset$), tak aj $\bigcap_{i \in I} A_i \in \mathcal{C}$.

Closed sets

Proposition

Let X be an arbitrary set. Let $\mathcal{C} \subseteq \mathcal{P}(X)$ be a system of sets satisfying the conditions (C1), (C2), (C3). Then

$$\mathcal{T} = \{X \setminus C; C \in \mathcal{C}\}$$

is a topology on X .

Moreover, the closed sets in $(X, \mathcal{T})$ are precisely the sets belonging to $\mathcal{C}$.

Cofinite and cocountable topology

$$\mathcal{T}_{cof} = \{\emptyset\} \cup \{X \setminus F; F \text{ is a finite subset of } X\}$$

$$\mathcal{T}_{coc} = \{\emptyset\} \cup \{X \setminus F; F \text{ is a countable subset of } X\}$$

Bases

Definition

Let $(X, \mathcal{T})$ be a topological space. A system $\mathcal{B} \subseteq \mathcal{T}$ is called a *basis for the topology* $\mathcal{T}$ if every open set U is a union of some system of sets from $\mathcal{B}$.

$$(\forall U \in \mathcal{T})(\exists \mathcal{S} \subseteq \mathcal{B}) U = \bigcup \mathcal{S}$$

- ▶ $\mathcal{B} \subseteq \mathcal{T}$, i.e., every basic set is open
- ▶ Equivalent condition: For any $x \in X$ and any open neighborhood $U \ni x$ there exists $B \in \mathcal{B}$ such that $x \in B \subseteq U$.

$$(\forall x \in U \in \mathcal{T})(\exists B \in \mathcal{B}) x \in B \subseteq U$$

Bases

Theorem

If $\mathcal{B}$ is a basis for some topology on X then:

(B1). $\mathcal{B}$ covers X , i.e.,

$$\bigcup \mathcal{B} = X.$$

(B2). *If $B_{1,2} \in \mathcal{B}$ both contain a point $x \in X$ then there exists $B \in \mathcal{B}$ with $x \in B \subseteq B_1 \cap B_2$.*

$$(\forall x \in X)[x \in B_{1,2} \in \mathcal{B} \Rightarrow (\exists B \in \mathcal{B})x \in B \subseteq B_1 \cap B_2]$$

Bases

Conversely, if $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfies (B1) and (B2) then the set of all unions of subsystems of $\mathcal{B}$ gives a topology $\mathcal{T}$ on X .

$$\mathcal{T} = \{\bigcup \mathcal{C}; \mathcal{C} \subseteq \mathcal{B}\}$$

Moreover, $\mathcal{B}$ is a basis for $\mathcal{T}$.

Examples of bases

- ▶ $\mathcal{B}_{disc} = \{\{x\}; x \in X\}$
- ▶ $\mathcal{B}_{ind} = \{X\}$

Topology given by a metric

$$B(a, r) = \{x \in X; d(x, a) < r\}$$

$$\mathcal{B} = \{B(a, r); a \in X, r \in \mathbb{R}, r > 0\}$$

Notation: $\mathcal{T}_d$

Another basis: $\mathcal{B}' = \{B(a, r); a \in X, r \in \mathbb{Q}, r > 0\}$

Sorgenfrey line

Sorgenfrey line (Lower limit topology)

$X = \mathbb{R}$ s topológíou danou bázou

$$\mathcal{B} = \{\langle a, b \rangle; a, b \in \mathbb{R}, a < b\}.$$

Notation: $\mathbb{R}_l$ or $\mathcal{T}_l$.

▶ $\langle a, b \rangle$ is clopen

▶ $\mathcal{T}_e \subseteq \mathcal{T}_l$

Subbasis

Definition

Nech $(X, \mathcal{T})$ je topológia a $\mathcal{S} \subseteq \mathcal{T}$. Hovoríme, že $\mathcal{S}$ *subbáza* topológie $\mathcal{T}$, ak

$$\mathcal{B} = \left\{ \bigcap \mathcal{F}; \mathcal{F} \subseteq \mathcal{S}, \mathcal{F} \text{ je neprázdna konečná množina} \right\}$$

tvorí bázu topológie $\mathcal{T}$.

T.j. $\mathcal{S}$ je subbáza práve vtedy, keď prieniky konečne veľa množín z $\mathcal{S}$ tvoria bázu.

Subbasis

Theorem

If $\mathcal{S}$ is a subbasis for some topology on X then:

(S1). $\mathcal{S}$ covers X , i.e.,

$$\bigcup \mathcal{S} = X.$$

Conversely, if $\mathcal{S} \subseteq \mathcal{P}(X)$ fulfills (S1) then the set of finite intersections of finite subsystems of $\mathcal{S}$, i.e.,

$$\mathcal{B} = \left\{ \bigcap \mathcal{F}; \mathcal{F} \subseteq \mathcal{S}, \mathcal{F} \text{ is a non-empty finite set} \right\}$$

is a basis for a topology on X .

Neighborhood

Definition

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. a subset $N \subseteq X$ is called a *neighborhood of the point* x if there exists an open set U with $x \in U \subseteq N$. If N is moreover an open set, we say that it is an *open neighborhood* of x .

We will denote by $\mathcal{N}_x$ the system of all neighborhoods of x , and by $\mathcal{O}_x$ the system of all open neighborhoods of x .

Neighborhood basis

Definition

Let $(X, \mathcal{T})$ be a topological space and $x \in X$. Let $\mathcal{B}_x \subseteq \mathcal{N}_x$, i.e., $\mathcal{B}_x$ is a system consisting of some neighborhoods of x . We say that $\mathcal{B}_x$ is a *neighborhood basis at the point x* if, for any open set U containing x , there exists $B \in \mathcal{B}_x$ with $B \subseteq U$.

$$(\forall U \in \mathcal{O}_x)(\exists B \in \mathcal{B}_x)x \in B \subseteq U$$

- ▶ Open balls are a neighborhood basis for $\mathcal{T}_d$.
- ▶ Closed intervals in $\mathbb{R}$ (closed balls in $\mathbb{R}^n$).

Neighborhood basis

Theorem

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B} \subseteq \mathcal{T}$ (i.e., $\mathcal{B}$ is a system of open sets). Then $\mathcal{B}$ is a base for X if and only if

$\mathcal{B}_x = \{B \in \mathcal{B}; x \in B\}$ is a neighborhood basis at x for each $x \in X$.

Sorgenfrey line

Example

$$\mathcal{B} = \{\langle a, b \rangle; a, b \in \mathbb{R}, a < b\}$$

$$\mathcal{B}_x = \{\langle a, b \rangle; a, b \in \mathbb{R}; a \leq x < b\}$$

$$\mathcal{B}'_x = \{\langle x, b \rangle; b \in \mathbb{R}; x < b\}$$

$$\mathcal{B}''_x = \{\langle x, b \rangle; b \in \mathbb{Q}; x < b\}$$

If x is irrational, the system

$$\mathcal{C}_x = \{\langle a, b \rangle; a, b \in \mathbb{Q}; a \leq x < b\}$$

is not a neighborhood basis at x .

Neighborhood basis

Theorem

Let $(X, \mathcal{T})$ be a topological spaces.

For every $x \in X$, let $\mathcal{B}_x$ be a neighborhood basis at $x \in X$ consisting only of open sets, i.e., $\mathcal{B}_x \subseteq \mathcal{T}$. Then the following holds:

(BO1). For each $B \in \mathcal{B}_x$ we have $x \in B$.

(BO2). If $U_{1,2} \in \mathcal{B}_x$ then there exists $U \in \mathcal{B}_x$ such that $U \subseteq U_1 \cap U_2$.

(BO3). if $y \in U \in \mathcal{B}_x$ then there exists $V \in \mathcal{B}_y$ such that $V \subseteq U$.

Neighborhood basis

Theorem

Conversely, suppose that for every point $x \in X$ we have a system $\mathcal{B}_x \subseteq \mathcal{P}(X)$ and that these systems fulfill the conditions (BO1)–(BO3). Then

$$\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$$

fulfills the conditions (B1) and (B2) and thus it is a basis for a topology $\mathcal{T}$ on the set X . Moreover, $\mathcal{B}_x$ is a neighborhood basis at x for this topology $\mathcal{T}$.

Moore plane

Example

On $\Gamma = \{(x_1, x_2) \in \mathbb{R}^2; x_2 \geq 0\}$ at the points $b = (b_1, b_2)$ with $b_2 > 0$ we take the basis given by the Euclidean metric, i.e.,
 $B(b, r) = \{(x_1, x_2) \in \Gamma; \|x - y\|_2 = \sqrt{(x_1 - b_1)^2 + (x_2 - b_2)^2} < r\}$,
 and for the points where the second coordinate is zero"

$$\mathcal{B}_{(b_1, 0)} = \{(b_1, 0)\} \cup \{\sqrt{(x_1 - b_1)^2 + (x_2 - r)^2} < r\}$$

Moore plane

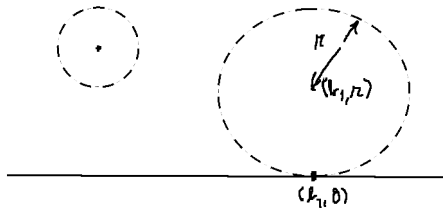


Figure: Bázové množiny v Mooreovej rovine.

Closure

Definition

Let $(X, \mathcal{T})$ be topological space and $A \subseteq X$. Then the set

$$\overline{A} = \bigcap \{C; A \subseteq C \subseteq X; C \text{ je uzavretá podmnožina } X\} \quad (1)$$

is called the *closure of the set A*.

Sometimes we use the notation $\text{cl}(A)$ or $\text{cl}_{\mathcal{T}}(A)$ – for example, if we want to work with closures of the same set in two different topologies.

Closure

Proposition

Nech $(X, \mathcal{T})$ je topologický priestor.

- (i) The set $\overline{A}$ is closed for any $A \subseteq X$.*
- (ii) A set $A \subseteq X$ is closed iff $A = \overline{A}$.*
- (iii) If $A \subseteq C \subseteq X$ and C is a closed set then $\overline{A} \subseteq C$.*

Proposition

Let $(X, \mathcal{T})$ be a topological space and A, B be subsets of X . If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.

$$A \subseteq B \quad \Rightarrow \quad \overline{A} \subseteq \overline{B} \quad (2)$$

Closure

Theorem

For the closure of sets in a topological space X we have:

$$(CL1). \quad \overline{\emptyset} = \emptyset;$$

$$(CL2). \quad A \subseteq \overline{A};$$

$$(CL3). \quad \overline{A \cup B} = \overline{A} \cup \overline{B}$$

$$(CL4). \quad \overline{\overline{A}} = \overline{A}$$

Conversely, if an operator $\overline{}: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ fulfills the conditions (CL1)–(CL4) and if we define

$$\mathcal{C} = \{A \subseteq X; \overline{A} = A\},$$

the the system $\mathcal{C}$ fulfills (C1)–(C3). (Thus the sets such that $A = \overline{A}$ are exactly the closed set of the topology $\mathcal{T}$ obtained by taking the complements of the sets from $\mathcal{C}$.)

Closure

- ▶ Description “from above” and “from below”
- ▶ In metric spaces: Closure = limits of sequences.
- ▶ In topological spaces: Closure = limits of nets.

Closure

Proposition

Let $(X, \mathcal{T})$ be a topological space, $x \in X$, $A \subseteq X$.

Then $x \in \overline{A}$ if and only if every neighborhood U of the point x intersects A .

$$x \in \overline{A} \quad \Leftrightarrow \quad (\forall U \in \mathcal{N}_x)(A \cap U \neq \emptyset)$$

Locally finite systems

Definition

Let $(X, \mathcal{T})$ be a topological space and $\mathcal{S}$ by a system of subsets of X . The system $\mathcal{S}$ is called *locally finite* if, for every point $x \in X$, there exists a neighborhood $U \ni x$ which intersects only finitely many sets from $\mathcal{S}$. (I.e., the set $\{S \in \mathcal{S}; S \cap U \neq \emptyset\}$ is finite.)

Locally finite systems

Theorem

Let $\{A_i; i \in I\}$ be a locally finite system

$$\overline{\bigcup_{i \in I} A_i} = \bigcup_{i \in I} \overline{A_i}.$$

Corollary

Union of a locally finite system of closed sets is a closed set.

Interior of a set

Definition

Nech $(X, \mathcal{T})$ je topologický priestor a $A \subseteq X$. Potom *vnútro množiny* A definujeme ako

$$\text{Int } A = \bigcup \{U \subseteq A; U \text{ je otvorená v } X\}.$$

Proposition

Nech $(X, \mathcal{T})$ je topologický priestor, A je podmnožina X . Potom:

$$\overline{A} = X \setminus \text{Int}(X \setminus A)$$

$$\text{Int } A = X \setminus \overline{X \setminus A}$$

Interior of a set

Proposition

Let $(X, \mathcal{T})$ be a topological space and $A, B \subseteq X$. Then we have:

- (i) If $A \subseteq B$ then $\text{Int } A \subseteq \text{Int } B$.*
- (ii) $\text{Int } A \subseteq A$*
- (iii) $\text{Int}(A \cap B) = \text{Int } A \cap \text{Int } B$*
- (iv) The set $\text{Int } A$ is open. For any open set $U \subseteq X$ we have $\text{Int } U = U$.*
- (v) $\text{Int}(\text{Int } A) = \text{Int } A$*

Dense sets

Definition

Let $(X, \mathcal{T})$ be a topological space. A subset $D \subseteq X$ is *dense* in X if $\overline{D} = X$, i.e., the closure of D is the whole space.

An equivalent characterization is that D intersects every non-empty open set.

$$(\forall U \in \mathcal{T} \setminus \{\emptyset\}) D \cap U \neq \emptyset$$

We can use elements from some basis $\mathcal{B}$ instead the whole topology $\mathcal{T}$ in this condition.

Dense sets

Proposition

Let $(X, \mathcal{T})$ be a topological space, $U, D \subseteq X$. If D is a dense set and U is an open set then

$$\overline{U \cap D} = \overline{U}.$$