

Initial and final topology

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Topological constructions

- ▶ subspace
- ▶ quotient space
- ▶ topological sum
- ▶ product space
- ▶ Generalization: Initial and final topology.

Initial topology

Definition

For each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $f_i: X \rightarrow X_i$ be a function. Then the *initial topology* $\mathcal{T}$ (on the set X) *with respect to the system of functions* $\{f_i; i \in I\}$ is the coarsest topology such that every $f_i: (X, \mathcal{T}) \rightarrow (X_i, \mathcal{T}_i)$ is continuous.

Initial topology

Proposition

Let $\mathcal{T}$ be the initial topology on X w.r.t. the system of mappings $\{f_i: X \rightarrow X_i, i \in I\}$. Then

$$\mathcal{S} = \{f_i^{-1}[U]; i \in I, U \in \mathcal{T}_i\}$$

is a subbase for the initial topology.

Given any subbases $\mathcal{S}_i$ of the topologies $\mathcal{T}_i$, the system

$$\mathcal{S}' = \{f_i^{-1}[U]; i \in I, U \in \mathcal{S}_i\}$$

is also a subbase for the initial topology.

Examples of initial topology

- ▶ embedding
- ▶ product space
- ▶ subspace of a product
- ▶ weak and weak*-topology

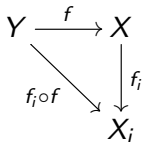
Continuity

Theorem

Let X be a set let $(X_i, \mathcal{T}_i)$ be a topological space for each $i \in I$.

Let $\mathcal{T}$ be the initial topology w.r.t. the system of mappings $\{f_i: X \rightarrow X_i, i \in I\}$.

For any topological space $(Y, \mathcal{T}')$ and any mapping $f: Y \rightarrow X$ we have: The mapping f is continuous if and only if the composition $f_i \circ f: Y \rightarrow X_i$ is continuous for every $i \in I$.



Initial topology and product

TODO

Final topology

Definition

Suppose that for each $i \in I$ we are given a topological space $(X_i, \mathcal{T}_i)$ and a function $f_i: X_i \rightarrow X$. Then the *final topology* $\mathcal{T}$ (on the set X) *with respect to the system of functions* $\{f_i; i \in I\}$ is the finest topology such that all functions $f_i: (X, \mathcal{T}) \rightarrow (X_i, \mathcal{T}_i)$ is continuous.

Final topology

Proposition

Suppose we have a system of functions $\{f_i: X_i \rightarrow X; i \in I\}$, where $(X_i, \mathcal{T}_i)$ is a topological space for every $i \in I$. Let us define

$$\mathcal{T} = \{U \subseteq X; (\forall i \in I) f_i^{-1}[U] \in \mathcal{T}_i\}.$$

Then $\mathcal{T}$ is a topology on X . This topology is the final topology w.r.t. this system of functions.

Examples of final topologies

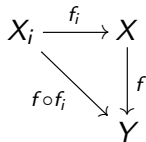
- ▶ faktorový priestor
- ▶ topologický súčet

$$X_i \xrightarrow{e_i} \coprod_{i \in I} X_i \xrightarrow{q} Y$$

Continuity

Theorem

For each $i \in I$, let $(X_i, \mathcal{T}_i)$ be a topological space and $\{f_i: X_i \rightarrow X; i \in I\}$ be a function. Let X as the final topology w.r.t. $\{f_i; i \in I\}$. Then a function $f: X \rightarrow Y$ is continuous if and only if $f \circ f_i$ is continuous for every $i \in I$.



Final topology

- ▶ The final topology w.r.t. the maps $f_i: X_i \rightarrow X$ is the same as the final topology w.r.t. $[f_i]: \coprod X_i \rightarrow X$.
- ▶ The function $[f_i]: \coprod X_i \rightarrow X$ is surjective iff $\bigcup_{i \in I} f[X_i] = X$.
- ▶ At the same time, the condition $\bigcup_{i \in I} f[X_i] = X$ describes when $[f_i]$ is a quotient map.