

General topology – assignments version 2

For each of the problems you can get maximum of 10 points. It is good to start new solution on a separate page. If you want to write the solutions in L^AT_EX, it will appreciate if I get both the source code and the pdf.

Úloha 1. For each $i \in I$, let X_i be a topological space and let $A_i \subseteq X_i$. Show that

$$\prod_{i \in I} \overline{A_i} = \overline{\prod_{i \in I} A_i}.$$

Úloha 2. Let $(D, \leq)$ be a directed set and $x: D \rightarrow \mathbb{R}$ satisfies

$$(\forall d, d' \in D)(d \leq d' \Rightarrow x_d \leq x_{d'}).$$

(I.e., it is a non-decreasing net consisting of real numbers.)

Show that if this net is bounded from above then there exists a limit $L = \lim_{d \in D} x_d$ and the equality

$$L = \sup_{d \in D} x_d$$

holds.

Úloha 3. Let A, B be disjoint closed subsets of a metric space (X, d) . Show that

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

defines a map $f: X \rightarrow \langle 0, 1 \rangle$ which is continuous and fulfills $f|_A = 0$ a $f|_B = 1$. (I.e., in metric spaces we can explicitly describe the map satisfying the conditions from Urysohn's lemma.)

Úloha 4. Show that if Urysohn's lemma holds for a topological space X then X is normal.

I.e., we want to show, that X is normal if, for any disjoint closed subsets A, B of X there exists a continuous function $f: X \rightarrow \langle 0, 1 \rangle$ such that

$$A \subseteq f^{-1}[\{0\}] \quad \text{a} \quad B \subseteq f^{-1}[\{1\}].$$

Úloha 5. Let $X = \{0, 1\}^\omega$ be the product of countably many copies of the discrete 2-point space and let $I = \langle 0, 1 \rangle$ be the closed unit interval with the usual topology. Let us define

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x_n}{2^{n+1}}.$$

a) Show that in this way we assigned to each $x \in X$ exactly one element $\varphi(x) \in I$. I.e., this is indeed $\varphi: X \rightarrow I$.

b) Show that the map φ is continuous.

c) Show that the map φ is surjective. Is it also injective?

d) Can you use these facts to prove compactness of $I = \langle 0, 1 \rangle$?